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Delay Systems with Limited Availability and Constant Holding Time

by MARTIN THIERER

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A delay system with n trunks is investigated. The connecting array s a grading with limited availability. The traffic offered has a Poisson distribution. The holding times are constant. The probability of delay and the mean waiting time are derived as functions of the traffic offered and compared with simulation test results.

Essential parts of this investigation were carried out at the Institute for Switching and Data Technics, University of Stuttgart, Germany.

Wartesysteme mit unvollkommener Erreichbarkeit und konstanter Belegungsdauer

Ein Wartesystem mit n abgehenden Leitungen wird behandelt. Die Koppelanordnung ist eine Mischung, d.h. die Abnehmerleitungen sind unvollkommen erreichbar. Der angebotene Verkehr hat eine Poisson-Verteilung. Die Belegungsdauern sind konstant. Die Wartewahrscheinlichkeit und die mittlere Wartezeit werden berechnet und mit Simulationstestergebnissen verglichen.

Wesentliche Teile dieser Untersuchung wurden am Institut für Nachrichtenvermittlung und Datenverarbeitung der Universität Stuttgart durchgeführt.

1. Introduction

Delay systems are found in modern centrally controlled switching systems and digital computers. The results of theoretical traffic analyses of these systems become increasingly important.

With the Interconnection Delay Formula (IDF) delay systems with limited availability can be analysed, if the holding times have a negatively exponential distribution [1]. The results of the Interconnection Delay Formula have been tabulated [2].

The central components of switching and computer systems are in most cases, however, occupied for a constant time. The following theory for delay systems with limited availability is developed under the assumption that the holding times are constant.

2. The Traffic

The offered traffic is assumed to be of a Poisson distribution of an average number of calls, c_A, offered per unit time. The number of traffic sources may be infinite. The probability that exactly r calls arrive during the interval t is given by

$$W_r(t) = \frac{(c_A t)^r}{r!} e^{-c_A t}. \tag{1}$$

All calls have the same constant service time h.

The traffic offered, A, is defined by

$$A = c_A h. \tag{2}$$

In delay systems the traffic offered is equal to the traffic carried, if no call is rejected and no call refuses to wait.

Therefore, the probability that exactly r calls arrive during the holding time h, is given by

$$W_r(h) = \frac{A^r}{r!} e^{-A}. \tag{3}$$

3. The System

A grading with n outgoing trunks is considered. The traffic is offered to g incoming groups. A call offered to an incoming group has access to k out of n outgoing trunks. The number k is called "availability". The trunk group of n outgoing trunks is a limited-access trunk group.

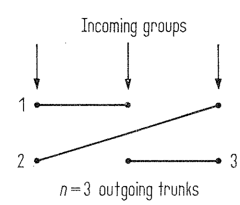

Fig. 1. Grading with $n = 3$, $k = 2$, $g = 3$.

In Fig. 1 an example of a grading with $n = 3$ outgoing trunks, the availability $k = 2$ and $g = 3$ incoming groups is shown.

A call offered to an incoming grading group is queuing in front of this group if all k trunks of this incoming group are busy.

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The number of waiting places is so large that all offered calls are able to wait.

If one occupation terminates on a trunk the calls are served in order of their arrival.

If the number of busy trunks x is greater than or equal to the availability k, a call offered to a grading will not always find an idle trunk. It is possible that all k trunks of the incoming group are busy and the call has to wait. The probability $c(x)$ that all k trunks in an incoming grading group are busy depends on the number of all busy trunks x in the grading.

If the probability $c(x)$ is known, the probability $u(x)$ that a call will find a free trunk in a grading is given by

$$u(x) = 1 - c(x) . \tag{4}$$

If the number of busy calls x is less than the availability k, a call offered to an incoming group will always find an idle trunk. The following equations hold:

$$c(x) = 0, \quad x = 0, 1, \ldots, k-1 . \qquad u(x) = 1 \tag{5}$$

4. The States

The state of a grading in delay systems may be characterized both by the number or busy trunks x and the sum z of calls waiting simultaneously in front of all incoming groups.

The probability of the state $\{x, z\}$ is denoted by $p(x, z)$.

If the number of busy trunks x is less than the availability k, no calls are waiting in front of an incoming group. Therefore it follows that

$$p(x, z) = 0, \quad x = 0, 1, \ldots, k-1; \quad z = 1, 2, \ldots . \tag{6}$$

The corresponding states $\{x, z\}$ are nonexistent.

5. The Probabilities of State

The probability of the state $\{i, j\}$ at the time t_0 is denoted by $p(i, j; t_0)$.

The theory will be evaluated in two steps.

Firstly, the probabilities $p(x, 0)$ are derived, secondly the probabilities $p(x, z)$.

5.1. The state $\{x, 0\}$

The system changes to the state $\{x, 0; t_0 + h\}$, if the state $\{0, 0; t_0\}$ exists at the time t_0 and x new calls arrive and seize free trunks during the interval h.

The transition probability that all of the x calls find a free trunk is given by

$$M(x) = u(0)\, u(1) \cdots u(x-1) = \prod_{l=0}^{x-1} u(l) . \tag{7}$$

The probability that
i) the state $\{0, 0; t_0\}$ exists at the time t_0,
ii) x new calls arrive during the interval h and
iii) all arriving calls find a free trunk
can be expressed in terms of eqs. (3) and (7):

$$p(0, 0; t_0)\, W_x(h)\, M(x) . \tag{8}$$

Assuming the system was in the arbitrary state $\{i, j; t_0\}$ at the time t_0. During the following intervall h, i calls terminate because their holding times h are constant. It may be assumed, that the j waiting calls and additionally $x - j$ new calls arrive at the system during the interval h. The system changes from the state $\{i, j; t_0\}$ to the new state $\{x, 0; t_0 + h\}$, if the number j is equal to or less than x, and if every waiting or arriving call finds a free trunk. The probability that the state $\{i, j; t_0\}$ changes to the state $\{x, 0; t_0 + h\}$ is equal to

$$p(i, j; t_0)\, W_{x-j}(h)\, M(x) , \quad j \leqq x . \tag{9}$$

All states changing to the state $\{x, 0; t_0 + h\}$ during the interval h are represented in Table 1. The corresponding probabilities are added in the right-hand column of Table 1.

Table 1. The transitions from $\{i, j; t_0\}$ to $\{x, 0; t_0 + h\}$.

State $\{i, j; t_0\}$	Number of arriving calls during h	Probability of transition from state $\{i, j; t_0\}$ to $\{x, 0; t_0 + h\}$
$\{0, 0\}$	x	$p(0, 0; t_0)\, W_x(h)\, M(x)$
$\{1, 0\}$	x	$p(1, 0; t_0)\, W_x(h)\, M(x)$
$\vdots$	$\vdots$	$\vdots$
$\{n, 0\}$	x	$p(n, 0; t_0)\, W_x(h)\, M(x)$
$\{k, 1\}$	$x-1$	$p(k, 1; t_0)\, W_{x-1}(h)\, M(x)$
$\vdots$	$\vdots$	$\vdots$
$\{n, 1\}$	$x-1$	$p(n, 1; t_0)\, W_{x-1}(h)\, M(x)$
$\vdots$	$\vdots$	$\vdots$
$\{k, x\}$	0	$p(k, x; t_0)\, W_0(h)\, M(x)$
$\vdots$	$\vdots$	$\vdots$
$\{n, x\}$	0	$p(n, x; t_0)\, W_0(h)\, M(x)$

Summing up the probabilities of Table 1 and using eq. (6) we have

$$p(x, 0; t_0 + h) = = M(x) \sum_{j=0}^{x} \left[W_{x-j}(h) \sum_{i=0}^{n} p(i, j; t_0) \right], \quad x = 0, 1, \ldots, n . \tag{10}$$

5.2. The state $\{x, z\}$

The system may be in the state $\{0, 0; t_0\}$ and r new calls may arrive during the interval h. It is possible that the system changes to the state $\{x, z; t_0 + h\}$, if the number r is equal to $x + z$. Other transitions are possible too, for instance, to the state $\{x + 1, z - 1; t_0 + h\}$. The transition to the state $\{x, z\}$ occurs with the transition probability $Ms(x, z)$.

The probability the state $\{0, 0; t_0\}$ turns to the state $\{x, z; t_0 + h\}$ equals

$$p(0, 0; t_0)\, W_{x+z}(h)\, Ms(x, z) . \tag{11}$$

Assuming the system was in the arbitrary state $\{i, j; t_0\}$ at the time t_0. During the following interval h, i calls terminate because the holding times h are constant. It may be assumed that the j waiting calls and additional $x + z - j$ new calls arrive at the system during the interval h. The system changes from the state $\{i, j; t_0\}$ to the state

$$\{x, z; t_0 + h\}$$

with the probability

$$p(i, j; t_0)\, W_{x+z-j}(h)\, Ms(x, z) , \quad j \leqq x + z . \tag{12}$$

Summing up all probabilities of the possible status $\{i, j; t_0\}$ and using eq. (6) we have

$$p(x, z; t_0 + h) = Ms(x, z) \sum_{j=0}^{x+z} \left[W_{x+z-j}(h) \sum_{i=0}^{n} p(i, j; t_0) \right], \quad x = 0, 1, \ldots, n; \quad z = 0, 1, \ldots . \tag{13}$$

Inserting $z = 0$ eq. (13) is identical to eq. (10).

Only the equilibrium state is investigated. Therefore, the probability distributions are independent of time:

$$p(x, z; t_0 + h) = p(x, z; t_0) = p(x, z) . \tag{14}$$

Under this condition eq. (13) becomes

$$p(x,z) = Ms(x,z) \sum_{j=0}^{x+z} \left[W_{x+z-j}(h) \sum_{i=0}^{n} p(i,j) \right], \quad x = 0,1,\dots,n; \quad z = 0,1,\dots. \tag{15}$$

Together with the condition

$$\sum_{x=0}^{n} \sum_{z=0}^{\infty} p(x,z) = 1 \tag{16}$$

one can calculate the probabilities $p(x, z)$.

To solve the equation system (15), (16) it is necessary to determine the probability $Ms(x, z)$.

6. The Probability $Ms(x, z)$

$Ms(x, z)$ is defined as the probability that $r = x + z$ arriving calls change the state $\{0, 0\}$ into the state $\{x, z\}$.

Assuming already $x + z - 1$ calls have arrived. The system has changed from state $\{0, 0\}$ to the state $\{x-1, z\}$ with the probability $Ms(x-1, z)$. The last call reaches a free incoming group with the probability $u(x-1)$. The number of busy trunks increases by 1. The system changes to the state $\{x, z\}$ with the probability

$$Ms(x-1,z)\, u(x-1)\,. \tag{17}$$

In another case the system has reached the state $\{x, z-1\}$ after $x + z - 1$ arriving calls with the probability $Ms(x, z-1)$. The last call arrives at an occupied incoming group with the probability $c(x)$. The call has to wait and the state of the system changes from $\{x, z-1\}$ to $\{x, z\}$ with the probability

$$Ms(x, z-1)\, c(x)\,. \tag{18}$$

All other states which the system has reached after $x + z - 1$ incoming calls do not lead to the state $\{x, z\}$. Therefore, it is possible to calculate the probability $Ms(x, z)$ from eqs. (17) and (18):

$$Ms(x,z) = Ms(x-1,z)\, u(x-1) + Ms(x,z-1)\, c(x), \quad x = 1,\dots,n; \quad z = 1,2,\dots. \tag{19}$$

Eq. (19) is a recurrence relation for the probability $Ms(x, z)$.

In the special case $z = 0$ the boundary values $M(x, 0)$ are equal to the probabilities $M(x)$ derived in eq. (7):

$$M(x,0) = M(x) = \prod_{l=0}^{x-1} u(l), \quad x = 0,1,\dots,n. \tag{20}$$

7. The Probability of Delay and the Mean Waiting Time

Let $p(x)$ be the probability of the state $\{x\}$, which means that x trunks of the outgoing trunk group are busy.

One obtains the probability $p(x)$ by the summation

$$p(x) = \sum_{z=0}^{\infty} p(x,z), \quad x = 0,1,\dots,n. \tag{21}$$

An arriving call will find x trunks busy with the probability $p(x)$. The probability that all trunks in the incoming group are busy was denoted by $c(x)$. One gets the probability of delay, W, by summation of all probabilities $p(x)\,c(x)$

$$W = \sum_{x=k}^{n} p(x)\, c(x)\,. \tag{22}$$

The waiting traffic Ω is defined as the average number of calls waiting simultaneously

$$\Omega = \sum_{z=1}^{\infty} z\, p(x,z)\,. \tag{23}$$

The waiting traffic is equal to the average number of delayed calls occuring per unit time, c_W, multiplied by the mean waiting time of the delayed calls t_W:

$$\Omega = c_W t_W\,. \tag{24}$$

The mean waiting time t_W divided by the mean holding time h may be noted by τ_W:

$$\tau_W = \frac{t_W}{h} = \frac{\Omega}{AW}\,. \tag{25}$$

8. Results of Simulation

A program was written for artificial traffic trials on a digital computer. The simulation results are shown in Fig. 2.

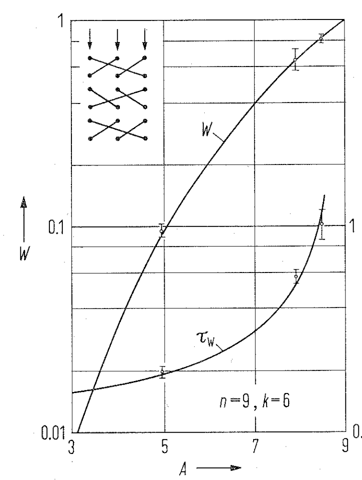

Fig. 2. The probability of delay W and the mean waiting time τ_W against the traffic offered A.

The test results of the probability of delay, W, and the mean waiting time, τ_W, are drawn in Fig. 2 together with the confidence intervals of 95%. The theoretical curves are calculated with eqs. (22) and (25). The calculation was performed with the probability

$$c(x) = \frac{\binom{x}{k}}{\binom{n}{k}} = \frac{x!\,(n-k)!}{(x-k)!\,n!} \tag{26}$$

introduced by A. K. Erlang [3]. This function of $c(x)$ yields exact results for Erlang's ideal gradings (Fig. 1). For non-ideal gradings (Fig. 2) eq. (26) is an approximation. The theoretical calculations are in good agreement with the simulation results.

References

[1] Thierer, M., Delay system with limited accessibility. 5. ITC New York 1967, Prebook, pp. 203–213.

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