# Exact Calculation of Overflow Systems with One Non-Ideal Grading and One Ideal Grading or Full Available Group

By **Rudolf Schehrer**

*A Report from the Institute for Switching and Data Technics, University of Stuttgart, Germany*)*

## 1. Introduction

In telephone networks, overflow systems with non-ideal gradings are used to a great extent. A very simple example of such a system is shown in *Fig. 1.* The telephone traffic $A$ is first offered to a primary trunk group with $n_1$ trunks and the availability $k_1$. If this group is blocked, calls can overflow to the secondary group with $n_2$ trunks and the availability $k_2$.

The exact calculation of such overflow systems leads to large sets of linear equations. This method is well known [6, 7]. If, e. g., the primary group as well as the secondary group are non-ideal gradings, a set of $2^{(n_1 + n_2)}$ linear equations must be solved. As this number of equations increases very rapidly with the total number of trunks $(n_1+n_2)$, this method can only be applied for very small overflow systems. If, however, one of the trunk groups is an ideal grading or a full available group, larger systems can also be calculated. Such systems will be considered in the following sections.

For systems consisting only of ideal gradings and/or full available groups the number of equations can be further reduced. Such overflow systems, however, are dealt with in a special paper [7] and will not be considered here.

## 2. Overflow Systems with a Non-ideal Primary Grading and an Ideal Secondary Grading

### 2.1. *The System*

This section deals with overflow systems consisting of a non-ideal primary grading and an ideal secondary grading as shown in *Fig. 2.*

For the following calculation method, equal offered overflow traffic to each secondary selector group is presumed. This condition is fulfilled if the uniform number of primary and secondary selector groups is determined as follows: Let $g'_1$ be the original number of primary selector groups and $g'_2$ that of the secondary selector groups. Then the uniform number of selector groups in the overflow system has to be

$$g = g'_1 \cdot g'_2 \tag{1}$$

where

$$g'_2 = \binom{n_2}{k_2}. \tag{2}$$

Primary and secondary selector groups have to be arranged such that each combination of a certain primary selector group and a certain secondary selector group occurs just once, as indicated schematically in Fig. 2.

### 2.2. *The Equations of State*

The trunks in the primary group must be numbered (in an arbitrary order). Then the grading of this group can be described by means of a matrix $M$ with the general element $m_{s,j}$ $(s=1, \ldots, k_1;$ $j=1, \ldots, g'_1)$. The matrix corresponding to the primary group in Fig 2 is shown in *Fig. 3.*

As the state congestion probabilities

$$\sigma_2(x_2) = \frac{\binom{x_2}{k_2}}{\binom{n_2}{k_2}}, \quad x_2 = 0, 1, \ldots, n_2 \tag{3a}$$

of the ideal secondary grading and the corresponding passage probabilities

$$\mu_2(x_2) = 1 - \sigma_2(x_2), \quad x_2 = 0, 1, \ldots, n_2 \tag{3b}$$

are known, it is not necessary to regard all its possible $2^{n_2}$ patterns of established calls; it is sufficient to consider the $(n_2+1)$ different global states.

For the description of the states „free“ and „busy“ of each individual trunk in the primary group a set of Boolean variables $z_i$ $(i=1, \ldots, n_1)$ with the following definition is used:

$z_i = 0$ if trunk No. $i$ is free,

$z_i = 1$ if trunk No. $i$ is busy.

The probability that the lines No. 1, 2, ..., $n_1$ of the primary group have a certain state $\{z_1, z_2, \ldots, z_{n1}\}$

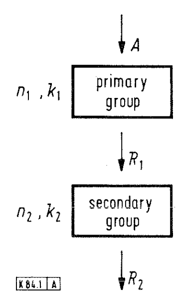



Fig. 1. Simple example of an overflow system.

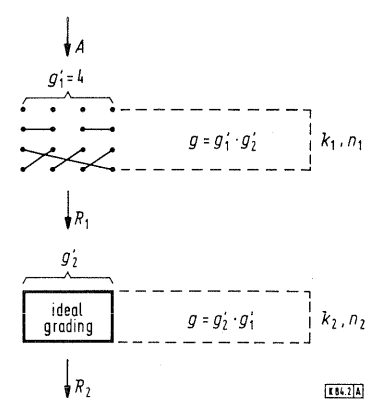
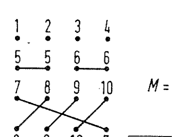



Fig. 2. Non-ideal primary grading with ideal secondary grading.




$$M = \left\|\begin{matrix} 1 & 2 & 3 & 4 \\ 5 & 5 & 6 & 6 \\ 7 & 8 & 9 & 10 \\ 8 & 9 & 10 & 7 \end{matrix}\right\|$$

Fig. 3. The matrix $M$.

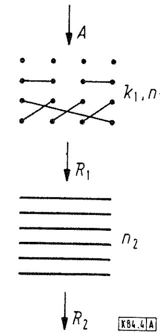



Fig. 4. Non-ideal primary grading with full available secondary group.

*) Author is now with AEG-Telefunken, Ulm

and that, furthermore, just $x_2$ trunks are busy in the secondary group is denoted by $p(z_1, z_2, \ldots, z_{n1}; x_2)$. Then the following equations of state are obtained according to the principle of statistical equilibrium:

$$\left[\sum_{i=1}^{n_1} z_i + x_2 + A \cdot \left(1 - \left[1 - \frac{1}{g'_1} \cdot \sum_{j=1}^{g'_1} \left(1 - \prod_{s=1}^{k_1} z_{m_{s,j}}\right)\right] \cdot \sigma_2(x_2)\right)\right] \cdot p(z_1, z_2, \ldots, z_{n_1}; x_2) =$$

$$= \sum_{i=1}^{n_1} (1 - z_i) \cdot p\,(z_1, z_2, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_{n_1}; x_2) + (1 - \alpha) \cdot (x_2 + 1) \cdot p(z_1, z_2, \ldots, z_{n_1}; x_2 + 1) +$$

$$+ \frac{A}{g'_1} \cdot \sum_{j=1}^{g'_1} \sum_{\varrho=1}^{k_1} \prod_{s=1}^{\varrho} z_{m_{s,j}} \cdot p(z_1, z_2, \ldots, z_{m_{\varrho,j}-1}, 0, z_{m_{\varrho,j}+1}, \ldots, z_{n_1}; x_2) + \tag{4a}$$

$$+ \beta \cdot \mu_2\,(x_2 - 1) \cdot \frac{A}{g'_1} \cdot \sum_{j=1}^{g'_1} \prod_{s=1}^{k_1} z_{m_{s,j}} \cdot p(z_1, z_2, \ldots, z_{n_1}; x_2 - 1)\,,$$

$$x_2 = 0, 1, \ldots, n_2,$$
$$z_i = 0, 1 \quad \text{for} \quad i = 1, 2, \ldots, n_1$$

where

$$\alpha = 1, \text{ if } x_2 = n_2, \text{ else } \alpha = 0,$$
$$\beta = 1 \text{ if } x_2 > 0\text{ , else } \beta = 0.$$

The product $\prod_{s=1}^{\varrho}$ refers to the first $\varrho$ outlets of a certain selector group. The summation $\sum_{s=1}^{k_1}$ comprises all busy-patterns of this selector group (column of the matrix $M$) where at least the first $\varrho$ outlets are busy.

The sum of all probabilities $p(z_1, z_2, \ldots, z_{n1}; x_2)$ is equal to one:

$$\sum_{z_1=0}^{1} \sum_{z_2=0}^{1} \cdots \sum_{z_{n_1}=0}^{1} \sum_{x_2=0}^{n_2} p(z_1, z_2, \ldots, z_{n1}; x_2) = 1\,. \tag{4b}$$

Solving such large sets of equations as (4a, b), it is useful to apply iterative methods, in particular the so-called successive overrelaxation method (SOR-method).

### 2.3. *The Loss Probabilities*

The overflow traffic $R_2$ (see Fig. 2) can easily be obtained from the state probabilities:

$$R_2 = \frac{A}{g'_1} \cdot \sum_{j=1}^{g'_1} \sum_{z_1=0}^{1} \sum_{z_2=0}^{1} \cdots \sum_{z_{n_1}=0}^{1} \sum_{x_2=k_2}^{n_2} \sigma_2(x_2) \times$$
$$\times\, p(z_1, z_2, \ldots, z_{n2}; x_2) \cdot \prod_{s=1}^{k_1} z_{m_{s,j}}\,. \tag{5}$$

Analogously the overflow traffic $R_1$ amounts to

$$R_1 = \frac{A}{g'_1} \cdot \sum_{j=1}^{g'_1} \sum_{z_1=0}^{1} \sum_{z_2=0}^{1} \cdots \sum_{z_{n_1}=0}^{1} \sum_{x_2=0}^{n_2} p(z_1, z_2, \ldots, z_{n1}; x_2) \times$$
$$\times \prod_{s=1}^{k_1} z_{m_{s,j}}\,. \tag{6}$$

With these values and with the offered traffic $A$ one can calculate easily the loss probability $B_1$ of the primary group, the loss $B_2$ of the secondary group and the total loss $B_{tot}$:

$$B_1 = R_1/A, \tag{7}$$
$$B_2 = R_2/R_1, \tag{8}$$
$$B_{tot} = R_2/A. \tag{9}$$

### 2.4. *Example*

For a primary grading, as shown in Fig. 2 and an ideal secondary grading with $n_2 = 10$ trunks and the availability $k_2 = 4$ the following values are obtained if a traffic of $A = 8$ Erlangs is offered:

$B_1 = 0.2108$, $R_1 = 1.6862$ Erlangs,
$B_2 = 0.0103$, $R_2 = 0.173$ Erlangs,
$B_{tot} = 0.00216$.

The number of selector groups according to Eqns. (1) and (2) is

$$g = 4 \cdot \binom{10}{4} = 840.$$

## 3. Overflow Systems with a Non-ideal Primary Grading and a Full Available Secondary Group

Systems of this kind represent the special case ($k_2 = n_2$) of the systems considered in Section 2 (see Fig. 4). Regarding that here

$$\sigma_2(x_2) = \begin{cases} 0 \text{ for } x_2 < n_2\,, \\ 1 \text{ for } x_2 = n_2\,, \end{cases} \tag{10}$$

the equations of state can be slightly simplified.

Besides its use in the calculation of overflow systems this method can be applied to the exact calculation of special gradings where the trunks connected to the $n_2$ last hunting steps form a full available subgroup.

This calculation method reduces the rank of the equation system to $2^{n_1} \cdot (n_2 + 1)$, instead of $2^{n_1} \cdot 2^{n_2}$ in the case of the method for general gradings.

### 3.1. *Example*

For a primary grading as shown in Fig. 4 with an offered traffic of $A = 8$ Erlangs and a full available secondary group of 10 trunks the values

$B_2 = 0.001378$, $R_2 = 0.002324$ Erlangs,
$B_{tot} = 0.000291$.

are obtained.

## 4. Overflow Systems with an Ideal Primary Grading and a Non-ideal Secondary Grading

In such an overflow system (as shown in *Fig. 5*), the primary group has the state congestion probabilities

$$\sigma_1(x_1) = \frac{\binom{x_1}{k_1}}{\binom{n_1}{k_1}}\,, \quad x_1 = 0,1,\ldots, n_1 \tag{11a}$$

and the passage probabilities

$$\mu_1(x_1) = 1 - \sigma_1(x_1), \quad x_1 = 0,\ 1,\dots, n_1. \tag{11b}$$

As explained in Section 2.1 the total number of selector groups has to be

$$g = g'_1 \cdot g'_2 . \tag{12a}$$

In this case we get

$$g = \binom{n_1}{k_1} \cdot g'_1 . \tag{12b}$$

Upon numbering the trunks of the secondary group, the states of these trunks and the matrix of the grading can be denoted as in Section 2.

Let the probability that just $x_1$ trunks are busy in the ideal primary grading and that furthermore the trunks No. 1, 2, ... $n_2$ of the secondary group are in a certain state $\{z_1, z_2, \dots, z_{n2}\}$ be denoted as $p(x_1; z_1, z_2, \dots, z_{n2})$. Then the following equations of state are obtained:

$$\begin{aligned}
&\left[x_1 + \sum_{i=1}^{n_2} z_i + A\cdot\mu_1(x_1) + \frac{A}{g'_2}\cdot\sigma_1(x_1) \times \right.\\
&\left.\times \sum_{j=1}^{g'_2}\left(1 - \prod_{s=1}^{k_2} z_{m_{s,j}}\right)\right]\cdot p(x_1; z_1, z_2, \dots, z_{n_2}) =\\
&= (1-\alpha)\cdot(x_1+1)\cdot p(x_1+1; z_1, z_2, \dots, z_{n_2}) +\\
&+ \sum_{i=1}^{n_2}(1-z_i)\cdot p(x_1; z_1, z_2, \dots, z_{i-1}, 1, z_{i+1}, \dots, z_{n_2}) +\\
&+ \beta\cdot A\cdot\mu_1(x_1-1)\cdot p(x_1-1); z_1, z_2, \dots, z_{n_2}) +\\
&+ \frac{A}{g'_2}\cdot\sigma_1(x_1)\cdot\sum_{j=1}^{g'_2}\sum_{\varrho=1}^{k_2}\prod_{s=1}^{\varrho} z_{m_{s,j}} \times\\
&\times p(x_1; z_1, z_2, \dots, z_{m_{\varrho,j}-1}, 0, z_{m_{\varrho,j}+1}, \dots, z_{n_2}),
\end{aligned} \tag{13a}$$

$$x_1 = 0,\ 1, \dots, n_1,$$
$$z_i = 0,\ 1 \text{ for } i = 1, 2, \dots, n_2.$$

where

$$\alpha = 1 \text{ if } x_1 = n_1, \text{ else } \alpha = 0,$$
$$\beta = 1 \text{ if } x_1 \neq 0\ , \text{ else } \beta = 0,$$

with the normalizing condition

$$\sum_{z_1=0}^{1}\sum_{z_2=0}^{1}\dots\sum_{z_{n_2}=0}^{1}\sum_{x_1=0}^{n_1} p(x_1; z_1, z_2, \dots, z_{n_2}) = 1. \tag{13b}$$

The SOR-method is suitable for solving these Eqns. (13). From the probabilities $p(x_1; z_1, z_2, \dots, z_{n2})$ the overflow traffic $R_2$ is obtained:

$$R_2 = \frac{A}{g'_2}\cdot\sum_{j=1}^{g'_2}\sum_{z_1=0}^{1}\sum_{z_2=0}^{1}\dots\sum_{z_{n_2}=0}^{1}\sum_{x_1=k_1}^{n_1}\sigma_1(x_1)\times$$
$$\times\ p(x_1; z_1, z_2, \dots, z_{n_2})\cdot\prod_{s=1}^{k_2} z_{m_{s,j}} . \tag{14}$$

The overflow traffic $R_1$ (and the loss $B_1$, resp.) can be determined according to Erlang's interconnection formula. Then the loss probabilities $B_2$ and $B_{tot}$ can be found with Eqns. (8) and (9).

### 4.1. *Example*

For an ideal primary grading with ($n_1 = 10$; $k_1 = 4$) and an secondary grading as shown in Fig. 5 one obtains the following values:

$B_1 = 0.19938,$
$R_1 = 1.595$ Erlangs,
$B_2 = 0.01079,$
$R_2 = 0.0172$ Erlangs,
$B_{tot} = 0.00215.$

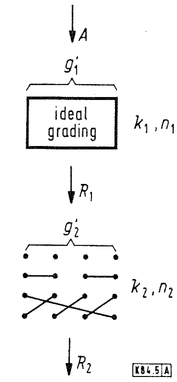



Fig. 5. Ideal primary grading with non-ideal secondary grading.

## 5. Overflow Systems with a Full Available Primary Group and a Non-ideal Secondary Grading

This type of system is a special case of the systems considered in Section 4. Since in this special case

$$\sigma_1(x_1) = \begin{cases} 0 \text{ for } x_1 \neq n_1 \\ 1 \text{ for } x_1 = n_1 \end{cases} \tag{15}$$

the equations of state (13a) can be simplified.

This kind of overflow system is realized very often in telephone networks with small (and therefore full available) primary groups and large final groups with limited access.

The investigation of such systems enables detailed studies about the effects of various grading structures on the loss probability in the case of offered overflow traffic.

### 5.1. *Example*

Let a random traffic of $A = 8$ Erlangs be offered to a full available primary group with $n_1 = 10$ trunks and a secondary grading as shown in Fig. 5. Then the following loss and overflow traffic values are obtained:

$B_1 = 0.12166,$ $R_1 = 0.973$ Erlangs,
$B_2 = 0.00992,$ $R_2 = 0.00965$ Erlangs,
$B_{tot} = 0.00121.$

## 6. Conclusion

In this paper, overflow systems with one non-ideal grading and one ideal grading or full available group are considered. For such systems exact calculation methods are derived. Besides their use in the calculation of overflow systems, these methods can also be applied to the exact calculation of special gradings.

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(Received by the NTZ: June 15, 1971)