ON THE EXACT CALCULATION OF OVERFLOW SYSTEMS

Rudolf Schehrer
Technical University Stuttgart
Stuttgart, Federal Republic of Germany

Sixth International Teletraffic Congress
Munich
September 9-15, 1970

ABSTRACT

Alternate routing systems with overflow facilities are widely used in long distance telephone networks. In such systems, calls can be switched via several routes. A very simple example of an alternate routing system, i. e. an overflow system, is shown in fig. 1. Calls which can not be

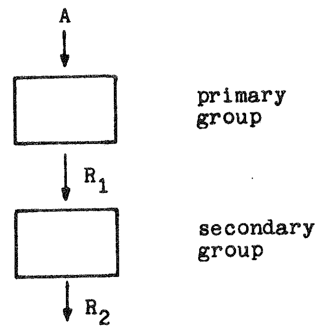


Fig. 1: Simple example of an overflow system

switched via the first route (primary group) overflow to the final route (secondary group).

The topic of this paper is the exact calculation of such overflow systems.

The primary group as well as the secondary group of an overflow system can either be a full available group, or an ideal grading, or a non-ideal grading. Thus there are 9 possible types of overflow systems. An exact solution is given for all of these types of systems in case of Poisson input, i. e. an infinite number of traffic sources. (Exact solutions which are already known will only be briefly referred to.)

Section 1 deals with systems consisting of two non-ideal gradings. Section 2 is concerned with overflow systems in which one group is an ideal grading or a full available group. In section 3, systems with ideal gradings and full available groups are considered.

Finally, section 4 treats overflow systems with two full available groups and a finite number of traffic sources.

1. OVERFLOW SYSTEMS CONSISTING OF NON-IDEAL GRADINGS

In telephone networks, overflow systems with non-ideal gradings are used to a great extent. A very simple example of such a system is shown in fig. 2. The primary group is a grading with

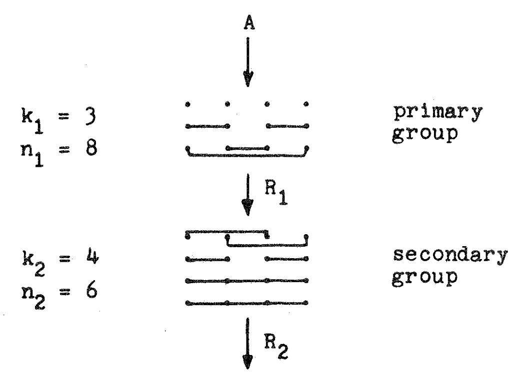


Fig. 2: Overflow system with two non-ideal gradings

$n_1 = 8$ trunks and the availability $k_1 = 3$. If this group is blocked, calls can overflow to the secondary group with $n_2 = 6$ trunks and the availability $k_2 = 4$.

For the exact solution of the loss B_1 in the primary group a set of 2^{n_1} linear equations must be solved. This method is well known [1, 2, 3]. Solving such large sets of equations, it is useful to apply iterative methods, in particular the so-called successive overrelaxation method (SOR-method).

The total loss B_{tot} (or the traffic R_2 overflowing behind the secondary group, resp.) can be calculated according to the same method. For this purpose the grading of the primary group and the grading of the secondary group can be regarded as one total grading with (n_1+n_2) trunks and the availability (k_1+k_2).

The loss B_2 in the secondary group is then found easily as

$$B_2 = \frac{B_{tot}}{B_1} \tag{1}$$

(From the probabilities of state of the total grading the loss B_1 can be calculated as well, so that only one set of equations must be solved.)

For the system shown in fig. 2 and an offered traffic of $A = 8$ Erlangs the following loss probabilities and overflow traffic values are obtained:

$B_1 = 0.3355$ $\quad R_1 = 2.684$ Erlangs
$B_2 = 0.1142$ $\quad R_2 = 0.307$ Erlangs
$B_{tot} = 0.0383$

The number of equations increases very rapidly with the number of trunks (n_1+n_2). (In the example mentioned above a set of $2^{14} = 16\,384$ equations is obtained.) Therefore this method can only be applied for very small overflow systems.

If, however, one or both groups are ideal gradings or full available groups, larger systems can also be calculated. Such systems will be considered in the following sections.

2. OVERFLOW SYSTEMS WITH ONE NON-IDEAL GRADING AND ONE IDEAL GRADING (OR FULL AVAILABLE GROUP)

2.1. Overflow Systems with a Non-ideal Primary Grading and an Ideal Secondary Grading

2.1.1. The System

This section deals with overflow systems consisting of a non-ideal primary grading and an ideal secondary grading as shown in fig. 3.

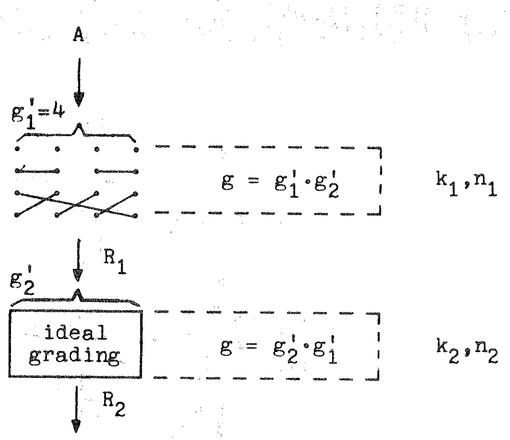


Fig. 3: Non-ideal primary grading with ideal secondary grading

For the following calculation method, equal offered overflow traffic to each secondary selector group is presumed. This condition is fulfilled if the uniform number of primary and secondary selector groups is determined as follows: Let g_1' be the original number of primary selector groups and g_2' that of the secondary selector groups. Then the uniform number of selector groups in the overflow system has to be

$$g = g_1' \cdot g_2' \tag{2}$$

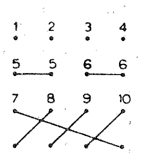
where

$$g_2' = \binom{n_2}{k_2} \tag{3}$$

Primary and secondary selector groups have to be arranged such that each combination of a certain primary selector group and a certain secondary selector group occurs just once, as indicated schematically in fig. 3.

2.1.2. The Equations of State

The trunks in the primary group must be numbered (in an arbitrary order). Then the grading of this group can be described by means of a matrix M with the general element $m_{s,j}$ ($s = 1..k_1$, $j = 1..g_1'$). The matrix corresponding to the primary group in fig. 3 is shown in fig. 4.



$$M = \begin{Vmatrix} 1 & 2 & 3 & 4 \\ 5 & 5 & 6 & 6 \\ 7 & 8 & 9 & 10 \\ 8 & 9 & 10 & 7 \end{Vmatrix}$$

Fig. 4: The matrix M

As the state congestion probabilities

$$\sigma_2(x_2) = \frac{\binom{x_2}{k_2}}{\binom{n_2}{k_2}}, \qquad x_2 = 0, 1, \dots, n_2 \tag{4a}$$

of the ideal secondary grading and the corresponding passage probabilities

$$\mu_2(x_2) = 1 - \sigma_2(x_2), \qquad x_2 = 0, 1, \dots, n_2 \tag{4b}$$

are known, it is not necessary to regard all its possible 2^{n_2} patterns of established calls; it is sufficient to consider the $(n_2 + 1)$ different global states.

For the description of the states "free" and "busy" of each individual trunk in the primary group a set of Boolean variables z_i ($i = 1..n_1$) with the following definition is used:

$z_i = 0$ if trunk No. i is free,
$z_i = 1$ if trunk No. i is busy.

The probability that the lines No. 1, 2, ... n_1 of the primary group have a certain state $\{z_1, z_2, \dots z_{n1}\}$ and that, furthermore, just x_2 trunks are busy in the secondary group is denoted by $p(z_1, z_2, \dots, z_{n1}; x_2)$. Then the following equations of state are obtained according to the principle of statistical equilibrium:

$$\begin{aligned}
&\left[\sum_{i=1}^{n_1} z_i + x_2 + A\cdot\left(1-\left[1-\frac{1}{g_1'}\sum_{j=1}^{g_1'}\Big(1-\prod_{s=1}^{k_1} z_{m_{s,j}}\Big)\right]\cdot\sigma_2(x_2)\right)\right]\cdot p(z_1, z_2, \dots, z_{n_1}; x_2) \\
&= \sum_{i=1}^{n_1}(1-z_i)\cdot p(z_1, z_2, \dots, z_{i-1}, 1, z_{i+1}, \dots, z_{n_1}; x_2) \\
&+ (1-\alpha)\cdot(x_2+1)\cdot p(z_1, z_2, \dots, z_{n_1}; x_2+1) \\
&+ \frac{A}{g_1'}\sum_{j=1}^{g_1'}\sum_{\varrho=1}^{k_1}\prod_{s=1}^{\varrho} z_{m_{s,j}}\cdot p(z_1, z_2, \dots, z_{m_{\varrho,j}-1}, 0, z_{m_{\varrho,j}+1}, \dots, z_{n_1}; x_2) \\
&+ \beta\cdot\mu_2(x_2-1)\cdot\frac{A}{g_1'}\sum_{j=1}^{g_1'}\prod_{s=1}^{k_1} z_{m_{s,j}}\cdot p(z_1, z_2, \dots, z_{n_1}; x_2-1),
\end{aligned} \tag{5a}$$

$$x_2 = 0, 1, \dots, n_2,$$
$$z_i = 0, 1 \quad \text{for } i = 1, 2, \dots, n_1$$

where

$\alpha = 1$, if $x_2 = n_2$, else $\alpha = 0$
$\beta = 1$ if $x_2 > 0$, else $\beta = 0$.

The product $\prod_{s=1}^{\varrho}$ in the fourth row of equation (5a) refers to the first ϱ outlets of a certain selector group. The summation $\sum_{s=1}^{k_1}$ comprises all busy-patterns of this selector group (column of the matrix M) where at least the first ϱ outlets are busy.

The sum of all probabilities $p(z_1, z_2, \dots, z_{n1}; x_2)$ is equal to one:

$$\sum_{z_1=0}^{1}\sum_{z_2=0}^{1}\dots\sum_{z_{n_1}=0}^{1}\sum_{x_2=0}^{n_2} p(z_1, z_2, \dots, z_{n_1}; x_2) = 1 \tag{5b}$$

For the solution of the equations (5a,b) the SOR-method is suitable.

2.1.3. The Loss Probabilities

The overflow traffic R_2 (see fig. 3) can easily be obtained from the state probabilities:

$$R_2 = \frac{A}{g_1'}\sum_{j=1}^{g_1'}\sum_{z_1=0}^{1}\sum_{z_2=0}^{1}\dots\sum_{z_{n_1}=0}^{1}\sum_{x_2=k_2}^{n_2}\sigma_2(x_2)\cdot p(z_1, z_2, \dots, z_{n_1}; x_2)\cdot\prod_{s=1}^{k_1} z_{m_{s,j}} \tag{6}$$

Analogously the overflow traffic R_1 amounts to

$$R_1 = \frac{A}{g_1'}\sum_{j=1}^{g_1'}\sum_{z_1=0}^{1}\sum_{z_2=0}^{1}\dots\sum_{z_{n_1}=0}^{1}\sum_{x_2=0}^{n_2} p(z_1, z_2, \dots, z_{n_1}; x_2)\cdot\prod_{s=1}^{k_1} z_{m_{s,j}} \tag{7}$$

With these values and with the offered traffic A one can calculate easily the loss probability B_1 of the primary group, the loss B_2 of the secondary group and the total loss B_{tot} :

$$B_1 = \frac{R_1}{A} \qquad (8)$$

$$B_2 = \frac{R_2}{R_1} \qquad (9)$$

$$B_{tot} = \frac{R_2}{A} \qquad (10)$$

2.1.4. Example

For a primary grading, as shown in fig. 3, and an ideal secondary grading with ($n_2 = 10$, $k_2 = 4$) the following values are obtained if a traffic of A = 8 Erlangs is offered:

B_1 = 0.2108 $\quad R_1$ = 1.6862 Erlangs

B_2 = 0.0103 $\quad R_2$ = 0.0173 Erlangs

B_{tot} = 0.00216

The number of selector groups according to eq. (2) and (3) is

$$g = 4\cdot\binom{10}{4} = 840$$

2.2. Overflow Systems with a Non-ideal Primary Grading and a Full Available Secondary Group

Systems of this kind represent the special case ($k_2 = n_2$) of the systems considered in section 2.1 (see fig. 5).

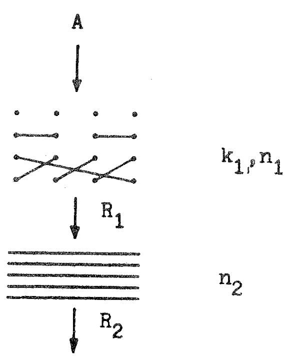


Fig. 5: Non-ideal primary grading with full available secondary group

Regarding that here

$$\sigma_2(x_2) = \begin{cases} 0 & \text{for } x_2 < n_2 \\ 1 & \text{for } x_2 = n_2 \end{cases} \qquad (11)$$

the equations of state can be slightly simplified.

Besides its use in the calculation of overflow systems this method can be applied to the exact calculation of special gradings where the trunks connected to the n_2 last hunting steps form a full available subgroup.

This calculation method reduces the rank of the equation system to $2^{n_1}\cdot(n_2+1)$, instead of $2^{n_1}\cdot 2^{n_2}$ in the case of the method for general gradings.

Example.

For a primary grading as shown in fig. 5 with an offered traffic of A = 8 Erlangs and a full available secondary group of 10 trunks the values

B_2 = 0.001378 , R_2 = 0.002324 Erlangs

B_{tot} = 0.000291

are obtained.

2.3. Overflow Systems with an Ideal Primary Grading and a Non-ideal Secondary Grading

In such an overflow system (as shown in fig. 6), the primary group has the state congestion probabilities

$$\sigma_1(x_1) = \frac{\binom{x_1}{k_1}}{\binom{n_1}{k_1}}, \quad x_1 = 0,1,\dots,n_1 \qquad (12a)$$

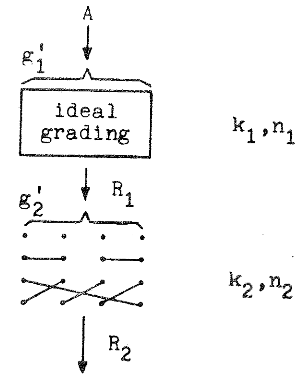


Fig. 6: Ideal primary grading with non-ideal secondary grading

and the passage probabilities

$$\mu_1(x_1) = 1 - \sigma_1(x_1), \quad x_1 = 0,1,\dots,n_1 \qquad (12b)$$

As explained in section 2.1.1 the total number of selector groups has to be

$$g = g_1' \cdot g_2' \qquad (13a)$$

In this case we get

$$g = \binom{n_1}{k_1}\cdot g_2' \qquad (13b)$$

Upon numbering the trunks of the secondary group, the states of these trunks and the matrix of the grading can be denoted as in section 2.1.1.

Let the probability that just x_1 trunks are busy in the ideal primary grading and that furthermore the trunks No. 1, 2, ... n_2 of the secondary group are in a certain state $\{z_1, z_2, \dots z_{n2}\}$ be denoted as $p(x_1;\, z_1, z_2, \dots, z_{n2})$. Then the following equations of state are obtained:

$$\begin{aligned}
&\Big[x_1 + \sum_{i=1}^{n_2} z_i + A\cdot\mu_1(x_1) + \frac{A}{g_2'}\cdot\sigma_1(x_1)\cdot\sum_{j=1}^{g_2'}\big(1-\prod_{s=1}^{k_2} z_{m_{s,j}}\big)\Big]\cdot p(x_1; z_1, z_2, \dots, z_{n_2}) \\
&= (1-\alpha)\cdot(x_1+1)\cdot p(x_1+1; z_1, z_2, \dots, z_{n_2}) \\
&+ \sum_{i=1}^{n_2} (1-z_i)\cdot p(x_1; z_1, z_2, \dots, z_{i-1}, 1, z_{i+1}, \dots, z_{n_2}) \\
&+ \beta\cdot A\cdot\mu_1(x_1-1)\cdot p(x_1-1; z_1, z_2, \dots, z_{n_2}) \\
&+ \frac{A}{g_2'}\cdot\sigma_1(x_1)\cdot\sum_{j=1}^{g_2'}\sum_{\varrho=1}^{k_2}\prod_{s=1}^{\varrho} z_{m_{s,j}}\cdot p(x_1; z_1, z_2, \dots, z_{m_{\varrho,j}-1}, 0, z_{m_{\varrho,j}+1}, \dots, z_{n_2}),
\end{aligned} \qquad (14a)$$

$$x_1 = 0,1,\dots,n_1 \;,$$
$$z_i = 0,1 \text{ for } i = 1,2,\dots,n_2$$

where

α = 1 if $x_1 = n_1$, else $\alpha = 0$

β = 1 if $x_1 \neq 0$, else $\beta = 0$

with the normalizing condition

$$\sum_{z_1=0}^{1}\sum_{z_2=0}^{1}\cdots\sum_{z_{n_2}=0}^{1}\sum_{x_1=0}^{n_1} p(x_1; z_1, z_2, \dots, z_{n_2}) = 1 \qquad (14b)$$

The SOR-method is suitable for solving these eq. (14a,b). From the probabilities $p(x_1;\, z_1, z_2, \dots, z_{n2})$ the overflow traffic R_2 is obtained:

$$R_2 = \frac{A}{g_2'}\cdot\sum_{j=1}^{g_2'}\sum_{z_1=0}^{1}\sum_{z_2=0}^{1}\cdots\sum_{z_{n_2}=0}^{1}\sum_{x_1=k_1}^{n_1}\sigma_1(x_1)\cdot p(x_1; z_1, z_2, \dots, z_{n_2})\cdot\prod_{s=1}^{k_2} z_{m_{s,j}} \qquad (15)$$

The overflow traffic R_1 (and the loss B_1, resp.) can be determined according to Erlang's interconnection formula. Then the loss probabilities B_2 and B_{tot} can be found with eq. (9) and (10).

Example.

For an ideal primary grading with ($n_1 = 10$; $k_1 = 4$) and a secondary grading as shown in fig. 6 one obtains the following values:

B_1 = 0.19938 $\quad R_1$ = 1.595 Erlangs

B_2 = 0.01079 $\quad R_2$ = 0.0172 Erlangs

B_{tot} = 0.00215

2.4. Overflow Systems with a Full Available Primary Group and a Non-ideal Secondary Grading

This type of system is a special case of the systems considered in section 2.3. Since in this special case

$$\sigma_1(x_1) = \begin{cases} 0 & \text{for } x_1 \neq n_1 \\ 1 & \text{for } x_1 = n_1 \end{cases} \tag{16}$$

the equations of state (14a) can be simplified.

This kind of overflow system is realized very often in telephone networks with small (and therefore full available) primary groups and large final groups with limited access.

The investigation of such systems enables detailed studies about the effects of various grading structures on the loss probability in the case of offered overflow traffic.

Example.

Let a random traffic of A = 8 Erlangs be offered to a full available primary group with $n_1 = 10$ trunks and a secondary grading as shown in fig. 6. Then the following loss and overflow traffic values are obtained:

$B_1 = 0.12166$ $R_1 = 0.973$ Erlangs
$B_2 = 0.00992$ $R_2 = 0.00965$ Erlangs
$B_{tot} = 0.00121$

3. OVERFLOW SYSTEMS WITH IDEAL GRADINGS AND FULL AVAILABLE GROUPS

3.1. Overflow Systems with Two Ideal Gradings

Such systems (as shown in fig. 7) can be calculated exactly by means of Bretschneider's

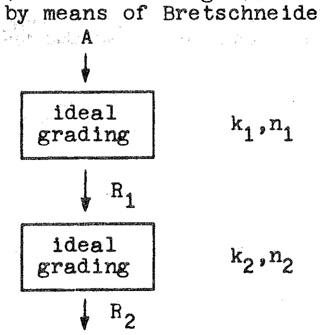


Fig. 7: Ideal primary grading with ideal secondary grading

method [4]. For comparison, however, the equations of state shall be mentioned briefly:

$$\begin{aligned}\left[x_1+x_2+A\cdot\left(1-\sigma_1(x_1)\cdot\sigma_2(x_2)\right)\right]\cdot p(x_1,x_2) \\ = \alpha\cdot(x_1+1)\cdot p(x_1+1,x_2) \\ +\beta\cdot(x_2+1)\cdot p(x_1,x_2+1) \\ +A\cdot\gamma\cdot\mu_1(x_1-1)\cdot p(x_1-1,x_2) \\ +A\cdot\delta\cdot\sigma_1(x_1)\cdot\mu_2(x_2-1)\cdot p(x_1,x_2-1)\,, \\ x_1 = 0,1,\dots,n_1 \\ x_2 = 0,1,\dots,n_2\end{aligned} \tag{17a}$$

where

$\alpha = 1$ if $x_1 < n_1$, else $\alpha = 0$
$\beta = 1$ if $x_2 < n_2$, else $\beta = 0$
$\gamma = 1$ if $x_1 > 0$, else $\gamma = 0$
$\delta = 1$ if $x_2 > 0$, else $\delta = 0$

with the normalizing condition

$$\sum_{x_1=0}^{n_1}\sum_{x_2=0}^{n_2} p(x_1,x_2) = 1 \tag{17b}$$

Here $p(x_1, x_2)$ means the probability that x_1 trunks are busy in the primary group and x_2 in the secondary group. The equations (17a,b) can be solved with the aid of the SOR-method.

Example.

For $n_1 = 10$, $k_1 = 4$, $n_2 = 10$, $k_2 = 4$, and an offered traffic of A = 8 Erlangs one obtains

$B_2 = 0.009506$ $R_2 = 0.01516$ Erlangs
$B_{tot} = 0.001895$

3.2. Overflow Systems with an Ideal Primary Grading and a Full Available Secondary Group

Overflow systems of this kind are a special case of the systems mentioned in section 3.1. Regarding, however, eq. (11), the formulae mentioned in section 3.1 can be slightly simplified.

Example.

For $n_1 = 10$, $k_1 = 4$, $n_2 = 10$, and an offered traffic of A = 8 Erlangs one obtains the values

$B_2 = 0.001278$ $R_2 = 0.00204$ Erlangs
$B_{tot} = 0.000255$

3.3. Overflow Systems with a Full Available Primary Group and an Ideal Secondary Grading

3.3.1. The System and the Equations of State

Such systems (as shown in fig.8) could be calculated according to the method mentioned in

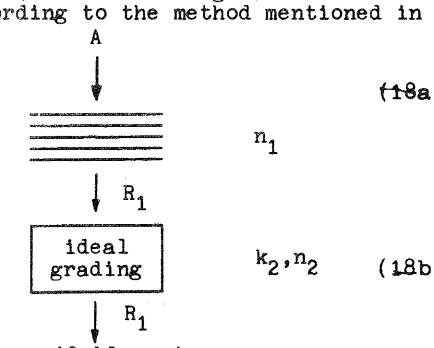


Fig. 8: Full available primary group with ideal secondary grading

section 3.1. If, however, eq. (16) is taken into account, the equations of state can be simplified remarkably, and a new type of equation is obtained:

$$\begin{aligned}(x_1+x_2+A)\cdot p(x_1,x_2) = \ & (x_1+1)\cdot p(x_1+1,x_2) \\ & +(x_2+1)\cdot p(x_1,x_2+1) \\ & + A\cdot p(x_1-1,x_2)\,, \\ & x_1 = 0,1,\dots,n_1-1\,, \\ & x_2 = 0,1,\dots,n_2\end{aligned} \tag{18a}$$

$$\begin{aligned}\left[n_1+x_2+A\mu(x_2)\right]\cdot p(n_1,x_2) = \ & (x_2+1)\cdot p(n_1,x_2+1) \\ & + A\cdot p(n_1-1,x_2) \\ & + A\mu(x_2-1)\cdot p(n_1,x_2-1)\,, \\ & x_2 = 0,1,\dots,n_2\end{aligned} \tag{18b}$$

with the normalizing condition

$$\sum_{x_1=0}^{n_1}\sum_{x_2=0}^{n_2} p(x_1,x_2) = 1 \tag{18c}$$

and with

$$p(x_1,x_2) = 0 \quad \text{for} \quad \begin{aligned} & x_1 < 0, \\ & x_2 < 0, \\ & x_1 > n_1, \\ & x_2 > n_2 \end{aligned} \tag{18d}$$

It should be pointed out that in (18a) there is no term corresponding to the last term in eq. (17a).

In the following, an explicit solution of the equations (18a,b,c) will be derived.

3.3.2. Graphic Illustration of the Equations of State

In the following schemes (fig. 9, 10, 11, 12) each of the probabilities $p(x_1, x_2)$ is represented by a crosspoint of a grid as shown in fig. 9.

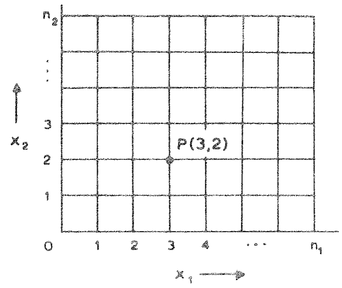


Fig. 9: Representation of the probabilities $p(x_1, x_2)$ by the crosspoints of a grid

The eq. (18a,b) connect 2, 3, or 4 probabilities $p(x_1, x_2)$ each. These equations are represented in fig. 10, 11, 12 by small graphs which connect the points corresponding to the p-values. Fig. 10 shows the graph representing eq. (18a)

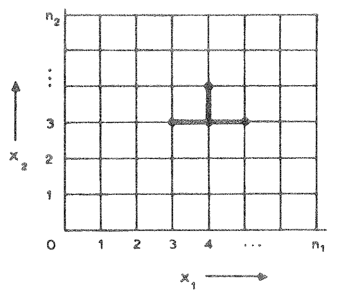


Fig. 10: Illustration of eq. (18a) for $x_1 = 4$, $x_2 = 3$

for $x_1 = 4$, $x_2 = 3$. In fig. 11 some more examples of eq. (18a) are indicated, including the

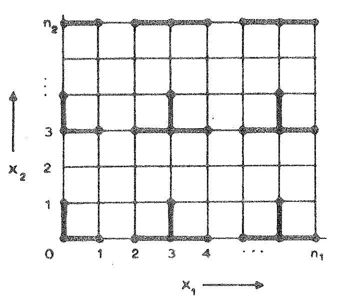


Fig. 11: Illustration of eq. (18a)

special cases $x_1 = 0$, $x_2 = 0$, and $x_2 = n_2$. Fig. 12 shows examples corresponding to eq. (18b).

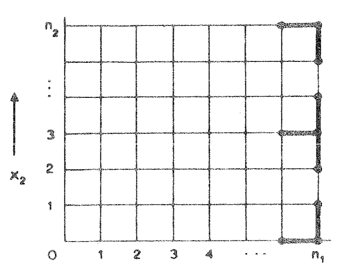


Fig. 12: Illustration of eq. (18b)

3.3.3. Reduction to a One-dimensional System

In a first step, the probabilities $p(x_1, x_2)$ can be expressed as a function of the values $p(0, x_2)$ only.

For $x_1 = 0$ and $x_2 = n_2$ eq. (18a) yields a relation between $p(0, n_2)$ and $p(1, n_2)$ as indicated in the upper left corner of fig. 11. Thus, if the probability $p(0, n_2)$ is given, also $p(1, n_2)$ can be calculated.

For $x_1 = 1$ and $x_2 = n_2$, eq. (18a) constitutes a relation between $p(0, n_2)$, $p(1, n_2)$, and $p(2, n_2)$. Therefore, knowing $p(0, n_2)$ and $p(1, n_2)$, the pr. $p(2, n_2)$ can be determined, and so forth.

Obviously all probabilities $p(x_1, n_2)$ are functions of the pr. $p(0, n_2)$ only.

For $x_1 = 0$, $x_2 = n_2 - 1$ eq. (18a) contains the pr. $p(0, n_2)$, $p(0, n_2 - 1)$, and $p(1, n_2 - 1)$. Thus, if also $p(0, n_2 - 1)$, besides $p(0, n_2)$, is given, the pr. $p(1, n_2 - 1)$ can be evaluated. With eq. (18a) for ($x_1 = 1$; $x_2 = n_2 - 1$) the pr. $p(2, n_2 - 1)$ can be obtained, etc.

Thus, all pr. $p(x_1, x_2)$ can be expressed as a function of only the $(n_2 + 1)$ probabilities $p(0,0)$, $p(0,1)$, ..., $p(0, x_2)$, ..., $p(0, n_2)$, which are situated at the left edge of the grid:

$$p(x_1, x_2) = f_{x_1, x_2}\left[p(0,0), \dots, p(0, x_2), \dots, p(0, n_2)\right] \tag{19}$$

If these functions f_{x_1, x_2} according to eq. (19) are calculated along the method shown above, the following expression is found:

$$p(x_1, x_2) = \sum_{\xi = x_2}^{n_2} (-1)^{\xi - x_2} \cdot \binom{\xi}{x_2} \cdot S_{\xi, x_1 + x_2 - \xi} \cdot p(0, \xi) \tag{20a}$$

$$x_1 = 1, 2, \dots, n_1 \qquad x_2 = 0, 1, \dots, n_2$$

where

$$S_{r,m} = \sum_{v=0}^{m} \frac{A^{m-v}}{(m-v)!} \cdot \binom{r-1+v}{v}, \qquad r, m \geq 0 \tag{20b}$$

The formula (20a,b) holds true also in the special case of a *full available* secondary group (instead of an *ideal* secondary *grading* considered here). In connection with the calculation of such overflow systems with *full* available secondary groups (see section 3.4), formula (20a,b) has already been derived by E. Brockmeyer [5].

The same formula is obtained in the more general case considered here, because the state congestion probabilities $\sigma_2(x_2)$ of the secondary group, which make the difference between these systems, do not occur in eq. (18a) and (20a,b).

3.3.4. Reduction to the Probability $p(0, n_2)$

In this section it will be shown that the pr. $p(0, x_2)$ can be expressed by the pr. $p(0, n_2)$ with the aid of eq. (18b):

Inserting eq. (20a) into (18b), one obtains

$$\begin{aligned} & A \cdot \mu(x_2 - 1) \cdot \sum_{\xi = x_2 - 1}^{n_2} (-1)^{\xi - x_2 + 1} \binom{\xi}{x_2 - 1} \cdot S_{\xi, n_1 + x_2 - 1 - \xi} \cdot p(0, \xi) \\ = & \left[A \cdot \mu(x_2) + n_1 + x_2\right] \sum_{\xi = x_2}^{n_2} (-1)^{\xi - x_2} \binom{\xi}{x_2} \cdot S_{\xi, n_1 + x_2 - \xi} \cdot p(0, \xi) \\ & - A \cdot \sum_{\xi = x_2}^{n_2} (-1)^{\xi - x_2} \cdot \binom{\xi}{x_2} \cdot S_{\xi, n_1 - 1 + x_2 - \xi} \cdot p(0, \xi) \\ & - (x_2 + 1) \cdot \sum_{\xi = x_2 + 1}^{n_2} (-1)^{\xi - x_2 - 1} \cdot \binom{\xi}{x_2 + 1} \cdot S_{\xi, n_1 + x_2 + 1 - \xi} \cdot p(0, \xi), \end{aligned} \tag{21}$$

$$x_2 = 0, 1, \dots, n_2$$

or, substituting x_2 by $(x_2 + 1)$,

$$\begin{aligned} p(0, x_2) = & \frac{1}{A \cdot \mu(x_2) \cdot S_{x_2, n_1}} \cdot \sum_{\xi = x_2 + 1}^{n_2} (-1)^{\xi - x_2 - 1} \cdot p(0, \xi) \cdot \\ & \cdot \Big(\left[\mu(x_2) \cdot \binom{\xi}{x_2} - \binom{\xi}{x_2 - 1}\right] \cdot A \cdot S_{\xi, n_1 + x_2 - \xi} \\ & + \left[A \cdot \mu(x_2 + 1) + n_1 + x_2 + 1\right] \cdot \binom{\xi}{x_2 + 1} \cdot S_{\xi, n_1 + x_2 + 1 - \xi} \\ & + (x_2 + 1) \cdot \binom{\xi}{x_2 + 2} \cdot S_{\xi, n_1 + x_2 + 2 - \xi} , \end{aligned} \tag{22}$$

$$x_2 = 0, 1, \dots, n_2 - 1$$

From the eq. (18a) and (20a) follows for $x_2 = n_2$

$$(A+x_1+n_2)\cdot S_{n_2,x_1} = (x_1+1)\cdot S_{n_2,x_1+1} + A\cdot S_{n_2,x_1-1}$$

or, generally ($x_1 \to m$, $n_2 \to r$):

$$(A+m+r)\cdot S_{r,m} = (m+1)\cdot S_{r,m+1} + A\cdot S_{r,m-1} \tag{23}$$

Inserting eq. (23) into (22) for $r = \xi$ and $m = n_1+x_2+1-\xi$, and regarding that

$$(x_2+2)\cdot\binom{\xi}{x_2+2} = (\xi-x_2-1)\cdot\binom{\xi}{x_2+1} \tag{24}$$

one obtains

$$\begin{aligned} p(0,x_2) = \frac{1}{S_{x_2,n_1}}\cdot\sum_{\xi=x_2+1}^{n_2}(-1)^{\xi-x_2-1}\cdot p(0,\xi)\cdot & \Big[\binom{\xi}{x_2}\cdot S_{\xi,n_1+x_2-\xi} \\ & - \frac{\sigma(x_2+1)}{\mu(x_2)}\cdot\binom{\xi}{x_2+1}\cdot S_{\xi,n_1+x_2+1-\xi} \\ & + \frac{n_1+1}{A\cdot\mu(x_2)}\cdot\binom{\xi}{x_2+1}\cdot S_{\xi,n_1+x_2+2-\xi}\Big], \\ & x_2 = 0,1,\ldots,n_2-1 \end{aligned} \tag{25}$$

or

$$p(0,x_2) = \sum_{\xi=x_2+1}^{n_2} a_{x_2,\xi}\cdot p(0,\xi), \qquad x_2 = 0,1,\ldots,n_2-1 \tag{26a}$$

with the abbreviation

$$\begin{aligned} a_{x,z} = \frac{(-1)^{z-x-1}}{S_{x,n_1}}\cdot & \Big[\binom{z}{x}\cdot S_{z,n_1+x-z} \\ & - \frac{\sigma(x+1)}{\mu(x)}\cdot\binom{z}{x+1}\cdot S_{z,n_1+x+1-z} \\ & + \frac{n_1+1}{A\cdot\mu(x)}\cdot\binom{z}{x+1}\cdot S_{z,n_1+x+2-z}\Big] \end{aligned} \tag{27}$$

Written out in full, eq. (26a) reads

$$\begin{aligned} p(0,n_2-1) &= a_{n_2-1,n_2}\cdot p(0,n_2) \\ p(0,n_2-2) &= a_{n_2-2,n_2-1}\cdot p(0,n_2-1) + a_{n_2-2,n_2}\cdot p(0,n_2) \\ p(0,n_2-3) &= a_{n_2-3,n_2-2}\cdot p(0,n_2-2) + a_{n_2-3,n_2-1}\cdot p(0,n_2-1) \\ & \qquad + a_{n_2-3,n_2}\cdot p(0,n_2) \\ \ldots & \qquad \ldots\ldots \\ p(0,0) &= a_{0,1}\cdot p(0,1) + a_{0,2}\cdot p(0,2) + \ldots + a_{0,n_2}\cdot p(0,n_2) \end{aligned} \tag{26b}$$

With this set of equations, the values $p(0,x_2)$ can be easily expressed as a function of $p(0,n_2)$ only:

$$\begin{aligned} p(0,n_2-1) &= b_{n_2-1}\cdot p(0,n_2) \\ p(0,n_2-2) &= b_{n_2-2}\cdot p(0,n_2) \\ \ldots & \qquad \ldots \\ p(0,0) &= b_0\cdot p(0,n_2) \end{aligned} \tag{28a}$$

or

$$p(0,x_2) = b_{x_2}\cdot p(0,n_2), \quad x_2 = 0,1,\ldots,n_2 \tag{28b}$$

where

$$\begin{aligned} b_{n_2} &= 1, \\ b_{n_2-1} &= a_{n_2-1,n_2} \\ b_{n_2-2} &= a_{n_2-2,n_2} + a_{n_2-2,n_2-1}\cdot a_{n_2-1,n_2} \\ b_{n_2-3} &= a_{n_2-3,n_2} + a_{n_2-3,n_2-1}\cdot a_{n_2-1,n_2} \\ & \quad + a_{n_2-3,n_2-2}\cdot(a_{n_2-2,n_2} + a_{n_2-2,n_2-1}\cdot a_{n_2-1,n_2}) \\ \ldots & \qquad \ldots\ldots \end{aligned} \tag{29a}$$

or

$$b_{x_2} = \sum_{z_1=x_2+1}^{n_2} a_{x_2,z_1}\cdot\sum_{z_2=z_1+1}^{n_2} a_{z_1,z_2}\cdot\sum_{z_3=z_2+1}^{n_2} a_{z_2,z_3}\cdot\ldots\cdot\sum_{z_w=z_{w-1}+1}^{n_2} a_{z_{w-1},z_w}$$

$$\text{for } x_2 = 0,1,\ldots,n_2 \quad \text{with } w = n_2-x_2 \tag{29b}$$

In eq. (29b) all summations in the product of the right hand side, having a lower bound greater than n_2 must be substituted by unity.

From the eq. (19), (28) and (29) one obtains

$$p(x_1,x_2) = p(0,n_2)\cdot\sum_{\xi=x_2}^{n_2}(-1)^{\xi-x_2}\cdot\binom{\xi}{x_2}\cdot b_\xi\cdot S_{\xi,x_1+x_2-\xi}, \qquad x_1 = 0,1,\ldots n_1, \quad x_2 = 0,1,\ldots n_1 \tag{30}$$

3.3.5. The State Probabilities

The value $p(0,n_2)$ can be determined by means of eq. (18c), or, more conveniently, from the equation

$$\sum_{x_2=0}^{n_2} p(x_1,x_2) = p_1(x_1) = \frac{\frac{A^{x_1}}{x_1!}}{\sum_{\xi=0}^{n_1}\frac{A^\xi}{\xi!}} = \frac{S_{0,x_1}}{S_{1,n_1}}, \qquad x_1 = 0,1,\ldots,n_1 \tag{31}$$

Using eq. (31) for $x_1 = 0$, one obtains

$$\sum_{x_2=0}^{n_2} p(0,x_2) = \frac{1}{\sum_{\xi=0}^{n_1}\frac{A^\xi}{\xi!}} = \frac{1}{S_{1,n_1}} \tag{32}$$

In serting eq. (28b) yields

$$p(0,n_2)\cdot\sum_{x_2=0}^{n_2} b_{x_2} = \frac{1}{S_{1,n_1}} = \frac{p_1(x_1)}{\frac{A^{x_1}}{x_1!}} \tag{33}$$

or

$$p(0,n_2) = \frac{p_1(x_1)}{\frac{A^{x_1}}{x_1!}\cdot\sum_{x_2=0}^{n_2} b_{x_2}} \tag{34}$$

From the eq. (30) and (34) the solution for the state probabilities $p(x_1,x_2)$ is obtained:

$$p(x_1,x_2) = p_1(x_1)\cdot\frac{\sum_{\xi=x_2}^{n_2}(-1)^{\xi-x_2}\cdot\binom{\xi}{x_2}\cdot b_\xi\cdot S_{\xi,x_1+x_2-\xi}}{\frac{A^{x_1}}{x_1!}\cdot\sum_{\xi=0}^{n_2} b_\xi}, \qquad x_1 = 0,1,\ldots,n_1, \quad x_2 = 0,1,\ldots,n_2 \tag{35}$$

The overflow traffic R_2 amounts to

$$R_2 = A\cdot\sum_{x_2=k_2}^{n_2}\sigma(x_2)\cdot p(n_1,x_2) \tag{36}$$

The loss B_1 (or the overflow traffic R_1, resp.) can be calculated according to Erlang's loss formula, the loss values B_2 and B_{tot} result from eq. (9) and (10).

3.3.6. Numerical Computation and Results

Using the relation

$$S_{r,m} = S_{r-1,m} + S_{r,m-1} \tag{37}$$

the S-polynomials can be evaluated successively by addition of two other values, starting with the numerical values

$$S_{r,0} = 1 \qquad \text{and} \qquad S_{0,m} = \frac{A^m}{m!}$$

Example.

For an overflow system with a full available primary group of $n_1 = 10$ trunks, an offered traffic of $A = 8$ Erlangs, and an ideal secondary grading of $n_2 = 10$ trunks and availability $k_2 = 4$ one obtains

$B_2 = 0.008889$ $\qquad R_2 = 0.008625$ Erlangs

$B_{tot} = 0.001081$

For calculating this example, the method shown here is faster by a factor 35 as compared with the SOR method.

3.3.7. Extension to Non-ideal Secondary Gradings

This method can also be applied to overflow systems with n o n - i d e a l secondary gradings (see section 2.4) if the state congestion probabilities $\sigma_2(x_2)$ are known with sufficient accuracy. For determining $G_2(x_2)$, the following approximation formula by U. Herzog and R. Kirsch[6] can be used:

$$G_2(x_2) = \frac{\binom{x_2}{k^*}}{\binom{n_2}{k^*}} \quad , \quad x_2 = 0,1,\dots,n_2 \tag{38}$$

with

$$k^* = k_2 - c_1 \cdot (k_2 - 5)\, \frac{n_2 - k_2}{n_2} \cdot \left(\frac{Y_2}{n_2} - c_2 \cdot \sqrt[4]{k_2 - 6} \right), \quad k_2 \geq 6 \tag{39}$$

where c_1 and c_2 are constants which can be found by a few simulation runs per grading type.

Example.

For an overflow system with $n_1 = 36$ trunks and a secondary group which consists of a so-called simplified standard grading (as shown in fig.13)

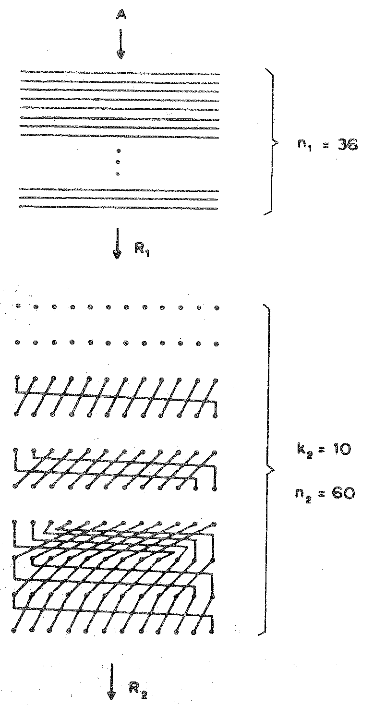


Fig. 13: Overflow system with a so-called simplified standard grading as secondary group

with $n_2 = 60$ trunks and the availability $k_2 = 10$ one obtains for an offered traffic of $A = 80$ Erlangs the value

$$B_2 = 0.0752$$

(For so-called simplified standard gradings of the German PTT the constants $c_1 = 1.26$ and $c_2 = 0.32$ are valid [6] . The traffic Y_2 carried in the secondary group was determined approximately for offered overflow traffic.)

A traffic simulation yields

$$B_2 = 0.0750 \pm 0.004$$

From this it can be seen that by means of this numerical method loss probabilities of high accuracy can be obtained even for remarkably large secondary groups.

3.4. Overflow Systems Consisting of Two Full Available Groups

The loss values in overflow systems where the primary group as well as the secondary group are full available can be easily obtained by applying Erlang's loss formula two times. The probability distribution in the secondary group can be calculated according to the well known analytic solution of E. Brockmeyer [5] .

5. OVERFLOW SYSTEMS WITH TWO FULL AVAILABLE GROUPS AND A F I N I T E NUMBER OF SOURCES

An overflow system consisting of 2 full available groups with a limited number of traf-

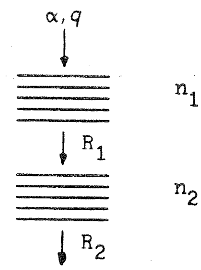


Fig. 14: Overflow system with full available groups and a finite number of sources

fic sources is shown in fig. 14. For this system the following equations of state are obtained:

$$\begin{aligned}[x_1 + x_2 + \alpha \cdot (q - x_1 - x_2)] \cdot p(x_1,x_2) = {} & \alpha \cdot [q - (x_1 - 1) - x_2] \cdot p(x_1 - 1, x_2) \\ & + (x_1 + 1) \cdot p(x_1 + 1, x_2) \\ & + (x_2 + 1) \cdot p(x_1, x_2 + 1) \end{aligned} \tag{40a}$$

$$x_1 = 0,1,\dots,n_1 - 1 \; ; \quad x_2 = 0,1,\dots,n_2$$

$$\begin{aligned}[n_1 + x_2 + \alpha \cdot \vartheta \cdot (q - n_1 - x_2)] \cdot p(x_1,x_2) = {} & \alpha \cdot [q - (n_1 - 1) - x_2] \cdot p(n_1 - 1, x_2) \\ & + \alpha \cdot [q - n_1 - (x_2 - 1)] \cdot p(n_1, x_2 - 1) \\ & + \vartheta (x_2 + 1) \cdot p(n_1, x_2 + 1) \\ & x_2 = 0,1,\dots,n_2 \end{aligned} \tag{40b}$$

where $\vartheta = 1$ if $x_2 < n_2$, else $\vartheta = 0$,

$$\sum_{x_1=0}^{n_2} \sum_{x_2=0}^{n_2} p(x_1,x_2) = 1 \tag{40c}$$

An analytic solution has been derived also for this type of overflow systems. For lack of space, however, this solution will be only briefly stated here.

The following expression for the probabilities $p(x_1,x_2)$, which is similar to eq. (19), can be derived with the aid of generating functions:

$$p(x_1,x_2) = \sum_{\xi = x_2}^{n_2} (-1)^{\xi - x_2} \cdot \binom{\xi}{x_2} \cdot T_{\xi, x_1 + x_2 - \xi} \cdot p(0,\xi) \tag{41}$$

where

$$T_{r,m} = \sum_{\eta = 0}^{m} \binom{q - \frac{\alpha \cdot r}{1+\alpha}}{m - \eta} \cdot \alpha^{m-\eta} \binom{\frac{r}{1+\alpha} - 1 + \eta}{\eta} \tag{42}$$

The value $p(n_1,n_2)$ can be calculated directly:

$$p(n_1,n_2) = \frac{\binom{q}{n_1 + n_2} \cdot \alpha^{n_1 + n_2}}{\sum_{\eta=0}^{n_1+n_2} \binom{q}{\eta} \cdot \alpha^{\eta}} \tag{43}$$

Furthermore, eq. (41) yields

$$p(n_1,n_2) = p(0,n_2) \cdot T_{n_2,n_1} \tag{44}$$

thus

$$p(0,n_2) = \frac{\binom{q}{n_1 + n_2} \cdot \alpha^{n_1 + n_2}}{T_{n_2,n_1} \cdot \sum_{\eta=0}^{n_1+n_2} \binom{q}{\eta} \cdot \alpha^{\eta}} \tag{45}$$

Inserting eq. (45) into (40b) leads to the relation

$$p(0,x_2) = \frac{1}{\alpha\cdot(q-n_1-x_2)\cdot T_{x_2,n_1}} \cdot \sum_{\xi=x_2+1}^{n_2} (-1)^{\xi-x_2-1} \cdot p(0,\xi) \cdot$$

$$\cdot \Big(\alpha\cdot(q-n_1-x_2)\cdot\Big[\binom{\xi}{x_2}-\binom{\xi}{x_2+1}\Big]\cdot T_{\xi,n_1+x_2-\xi} + \binom{\xi}{x_2+1}\cdot\big[n_1+x_2+\alpha\cdot\vartheta\cdot(q-n_1-x_2-1)\big]\cdot T_{\xi,n_1+x_2-\xi+1} + \vartheta\cdot(x_2+2)\cdot\binom{\xi}{x_2+2}\cdot T_{\xi,n_1+x_2-\xi+2}\Big), \quad x_2 = 0,1,\dots,n_2-1 \tag{46}$$

where $\vartheta = 1$ if $x_2 < n_2 - 1$, else $\vartheta = 0$, or in a simplified notation

$$p(0,x_2) = \sum_{\xi=x_2+1}^{n_2} a^*_{x_2,\xi}\cdot p(0,\xi), \quad x_2 = 0,1,\dots,n_2-1 \tag{47}$$

with the abbreviation

$$a^*_{x,z} = \frac{(-1)^{z-x-1}}{\alpha\cdot(q-n_1-x)\cdot T_{x,n_1}}\cdot\Big(\alpha\cdot(q-n_1-x)\cdot\Big[\binom{z}{x}-\binom{z}{x+1}\Big]\cdot T_{z,n_1+x-z} + \binom{z}{x+1}\cdot\big[n_1+x+\alpha\cdot\vartheta\cdot(q-n_1-x-1)\big]\cdot T_{z,n_1+x-z+1} + \vartheta\cdot(x+2)\cdot\binom{z}{x+2}\cdot T_{z,n_1+x-z+2}\Big), \tag{48}$$

where $\vartheta = 1$ if $x < n_2 - 1$, else $\vartheta = 0$.

Knowing $p(0,n_2)$ from eq. (45), all values $p(0,x_2)$ can be calculated successively according to eq. (47). Then the probabilities $p(x_1,x_2)$ can be easily evaluated by means of eq. (41).

This results in the following formulae for the probabilities $p(x_1,x_2)$:

$$p(x_1,x_2) = \frac{\binom{q}{n_1+n_2}\cdot\alpha^{n_1+n_2}}{T_{n_2,n_1}\cdot\sum_{\eta=0}^{n_1+n_2}\binom{q}{\eta}\alpha^{\eta}}\cdot\sum_{\xi=x_2}^{n_2}(-1)^{\xi-x_2}\cdot b^*_{\xi}\cdot\binom{\xi}{x_2}\cdot T_{\xi,x_1+x_2-\xi} \tag{49}$$

where

$$b^*_{x_2} = \sum_{z_1=x_2+1}^{n_2} a_{x_2,z_1}\cdot\sum_{z_2=z_1+1}^{n_2} a_{z_1,z_2}\cdot\sum_{z_3=z_2+1}^{n_2} a_{z_2,z_3}\cdot\ldots\cdot\sum_{z_w=z_{w-1}+1}^{n_2} a_{z_{w-1},z_w}$$

$$\text{for } x_2 = 0,1,\dots,n_2 \quad \text{with } w = n_2 - x_2 \tag{50}$$

Further one obtains the traffic Y_1 carried in the primary group and the traffic Y_2 carried in the secondary group,

$$Y_1 = \sum_{x_1=0}^{n_1}\sum_{x_2=0}^{n_2} x_1\cdot p(x_1,x_2) \tag{51}$$

$$Y_2 = \sum_{x_1=0}^{n_1}\sum_{x_2=0}^{n_2} x_2\cdot p(x_1,x_2) \tag{52}$$

the offered traffic

$$A = \alpha\cdot(q - Y_1 - Y_2) \tag{53}$$

and the overflow traffics

$$R_1 = A - Y_1 \tag{54}$$

$$R_2 = R_1 - Y_2 \tag{55}$$

The loss values B_1, B_2, and B_{tot} result from eq. (8), (9), and (10).

For the variance V of the traffic Y_2 carried in the secondary group one obtains

$$V = \sum_{x_1=0}^{n_1}\sum_{x_2=0}^{n_2}(x_2 - Y_2)^2\cdot p(x_1,x_2) \tag{56}$$

Example.

For a full available primary group with $n_1 = 10$ trunks, a full available secondary group with $n_2 = 10$ trunks, and for $q = 40$ traffic sources with a call intensity $\alpha = 0.5$ each the following values are obtained:

		Y_1 = 8.748	Erlangs
		Y_2 = 4.505	Erlangs
		A = 13.373	Erlangs
B_1	= 0.3458	R_1 = 4.625	Erlangs
B_2	= 0.0260	R_2 = 0.120	Erlangs

LITERATURE

[1] Palm, C.: Calcul exact de la perte dans les groupes de circuits echelonnes. Ericsson Technics, 4, 41, 1936

[2] Kosten, L.: Über Sperrungswahrscheinlichkeiten bei Staffelschaltungen. ENT 14 (1937), p. 5-12

[3] Brockmeyer, E., Halstrøm, H., Jensen, A.: The life and works of A. K. Erlang. Acta Polytechnica Scandinavia (1960), No. 287

[4] Bretschneider, G.: Über eine neue Klasse einstufig erreichbarer idealer Anordnungen von Fernsprechleitungen. AEÜ 17 (1963), p. 69-74

[5] Brockmeyer, E.: The simple overflow problem in the theory of telephone traffic. Teleteknik 5 (1954), p. 361-374

[6] Herzog, U., Kirsch, R.: Sperrwahrscheinlichkeitsuntersuchungen an ein- und mehrstufigen Koppelanordnungen. Diploma thesis at the Institute for Switching and Data Technics of the University Stuttgart (1966)

[7] Wallström, B.: Congestion stusies in telephone systems with overflow facilities. Ericsson Technics 22 (1966), vol.3

[8] Schehrer, R.: Über die exakte Berechnung von Überlaufsystemen der Wählvermittlungstechnik. Ph. D. Thesis, University Stuttgart, 1969