TRANSACTIONS

of the

SIXTH PRAGUE CONFERENCE

on

INFORMATION THEORY,
STATISTICAL DECISION FUNCTIONS,
RANDOM PROCESSES

held at

Prague, from September 19 to 25, 1971

P. Kühn

ON A COMBINED DELAY AND LOSS SYSTEM
WITH DIFFERENT QUEUE DISCIPLINES

ON A COMBINED DELAY AND LOSS SYSTEM WITH DIFFERENT QUEUE DISCIPLINES

PAUL KÜHN

STUTTGART

1. INTRODUCTION

The main operation of stochastic service systems is the connection of service demands (calls) with service facilities (servers) where the interarrival and service times are generally governed by probabilistic laws.

According to the acceptance mechanism during a blocking period (all servers are busy) loss systems, delay systems, and combined delay and loss systems are distinguished. In loss systems no call will be accepted during a blocking period. In delay systems all calls will be accepted; they are ordered into a waiting line (queue) and wait until a free server is allocated to give service to them. In combined delay and loss systems there is only a limited number of waiting places available; calls can either be rejected at their arrival or they can be pushed out after an unsuccessful waiting time.

The service quality of a stochastic service system is influenced by several factors as system parameters (number of servers and waiting places), operational modes (hunting and queue disciplines), traffic input and service duration processes (distribution functions of interarrival and service times). Characteristics for the quality of service can be calculated analytically by means of queueing theory.

Applications of stochastic service systems are widely spread, especially in modern telephone exchanges, data networks, and computer systems.

2. STATEMENT OF THE PROBLEM

A multiserver combined delay and loss system is to investigate with respect to different hunting and queue disciplines. The service system has n fully accessible servers and s_1 waiting places. Calls arrive according to a Poisson distribution with

According to the Markovian assumptions the Kolmogorov-forward-equation holds for the probabilities of state [1]

$$(3) \qquad P'(t;i) = -q_i P(t;i) + \sum_{j \neq i} q_{ji} P(t;j), \quad i,j \in \Pi,$$

where q_{ji}, $j \neq i$, denotes the intensity (transition coefficient) for transition from state j into state i and $q_i = \sum_{j \neq i} q_{ij}$.

In the stationary case

$$(4) \qquad \lim_{t \to \infty} P(t;i) = p(i), \quad i \in \Pi,$$

the derivatives $P'(t;i)$, $i \in \Pi$, vanish and (3) turns to the steady state equations

$$(5a) \qquad q_i p(i) = \sum_{j \neq i} q_{ji} p(j), \quad i,j \in \Pi.$$

The linear homogeneous system of equations (5a) is completed by the normalizing condition

$$(5b) \qquad \sum_{i \in \Pi} p(i) = 1.$$

For single server systems and multiserver systems with identical servers extensive work has been done in the time-dependent solutions, c.f. Takács [2]. In this paper, the stationary case will be assumed only.

3.2. Sequential hunting of servers

In case of sequential hunting an incoming call which is neither allowed to wait nor rejected occupies a specified server, namely the first idle server in the group of servers from 1 to n. Fig. 1a shows the state space and the transition coefficients for the example of $n = 3$ servers. The termination rate of the fully occupied server group is abbreviated by μ, where

$$(6) \qquad \mu = \sum_{\nu=1}^{n} \varepsilon_\nu.$$

The total state space is built from two subspaces, a n-dimensional subspace which stands for the occupation pattern of servers, and an one-dimensional subspace representing the state of the queue.

arrival rate λ. The service times are negative exponentially distributed with termination rate ε_ν for the ν-th server, $\nu = 1(1)\,n$. Two different disciplines are assumed for occupation of an idle server:

i) sequential hunting,
ii) random hunting.

The queue disciplines are:

i) first-come, first-served service (D1),
ii) random order of service (D2),
iii) last-come, first-served service (D3)
   a) with push-out priority,
   b) without push-out priority.

Arriving calls are served immediately if there is at least one server idle. They are allowed to wait when all servers are busy and if there is at least one waiting place available. In case of the fully occupied service system an arriving call is rejected (D1, D2, D3b), or it is accepted and pushes out that waiting call which had arrived first (D3a).

Under the assumption of stationarity the probabilities of state will be investigated by means of the Kolmogorov-forward-equation. Derived characteristics, as probabilities for waiting and loss, traffic carried, mean queue length, and the mean waiting time will be calculated as well. The distribution functions of waiting time and their moments are studied uniformly on the basis of Kolmogorov-backward-type differential equations.

3. PROBABILITIES OF STATE

3.1. Random process of system states

Let $\xi(t)$ be a random pattern representing the occupation of servers and waiting places at time t. In the general case ξ is a vector of $(n+1)$ components

$$(1) \qquad \xi = (x_1, \ldots, x_\nu, \ldots, x_n; z),$$

where $x_\nu \in [0,1]$ denotes the state of server ν, $\nu = 1(1)\,n$; $z \in [0, s_1]$ represents the number of occupied waiting places. Be Π the set of all possible $(2^n + s_1)$ states. For the random process $\{\xi(t), t \geq 0\}$ a probability of state is defined by

$$(2) \qquad P(t;i) = \text{Prob} \{\xi(t) = i\}, \quad t \geq 0, \, i \in \Pi.$$

$$\mu \cdot p(x_1,\ldots,x_\nu,\ldots,x_n;0) = \lambda \cdot \sum_{\nu=1}^{n} x_\nu \, p(\ldots,x_\nu-1,\ldots;0)$$
$$\text{for } \prod_{\nu=1}^{n} x_\nu = 1,$$

$$\mu \cdot p(1,\ldots,1;z) = \lambda \cdot p(1,\ldots,1;z-1), \quad z=1(1)s_1,$$

(7b)

$$\sum_{x_1=0}^{1}\cdots\sum_{x_\nu=0}^{1}\cdots\sum_{x_n=0}^{1} p(x_1,\ldots,x_\nu,\ldots,x_n;0) + \sum_{z=1}^{s_1} p(1,\ldots,1;z) = 1.$$

(7c)

The solution for (7b) is obtained recursively

$$p(1,\ldots,1;z) = p(1,\ldots,1;0)\cdot\varrho^z, \quad z=0(1)s_1,$$

(8)

where $\varrho = \lambda/\mu$ (occupancy).

An explicit solution for (7a) is not known; this set of equations has to be solved numerically by an iterative method.

3.3. Random hunting of servers

Contrarily to the sequential hunting mode, an incoming call occupies in case of random hunting a server which is selected randomly out of all idle servers. If there are $x = \sum_{\nu=1}^{n} x_\nu$ servers occupied at the call arrival instant, the probability for service by a specified server is $1/(n-x)$, $x\in[0,n)$. In Fig. 1b the state space and the transition coefficients are shown for the 3-server example.

The set of steady state equations (7a) has to be replaced by (9a) while (7b) and (7c) remain unchanged:

$$\left(\lambda + \sum_{\nu=1}^{n} x_\nu \varepsilon_\nu\right) \cdot p(x_1,\ldots,x_\nu,\ldots,x_n;0) = \frac{\lambda}{n-x+1}\cdot\sum_{\nu=1}^{n} x_\nu \, p(\ldots,x_\nu-1,\ldots;0) +$$
$$+ \sum_{\nu=1}^{n}(1-x_\nu)\varepsilon_\nu \, p(\ldots,x_\nu+1,\ldots;0)$$
$$\text{for } \prod_{\nu=1}^{n} x_\nu = 0,$$

(9a)

$$\mu \cdot p(x_1,\ldots,x_\nu,\ldots,x_n;0) = \lambda \cdot \sum_{\nu=1}^{n} x_\nu \, p(\ldots,x_\nu-1,\ldots;0)$$
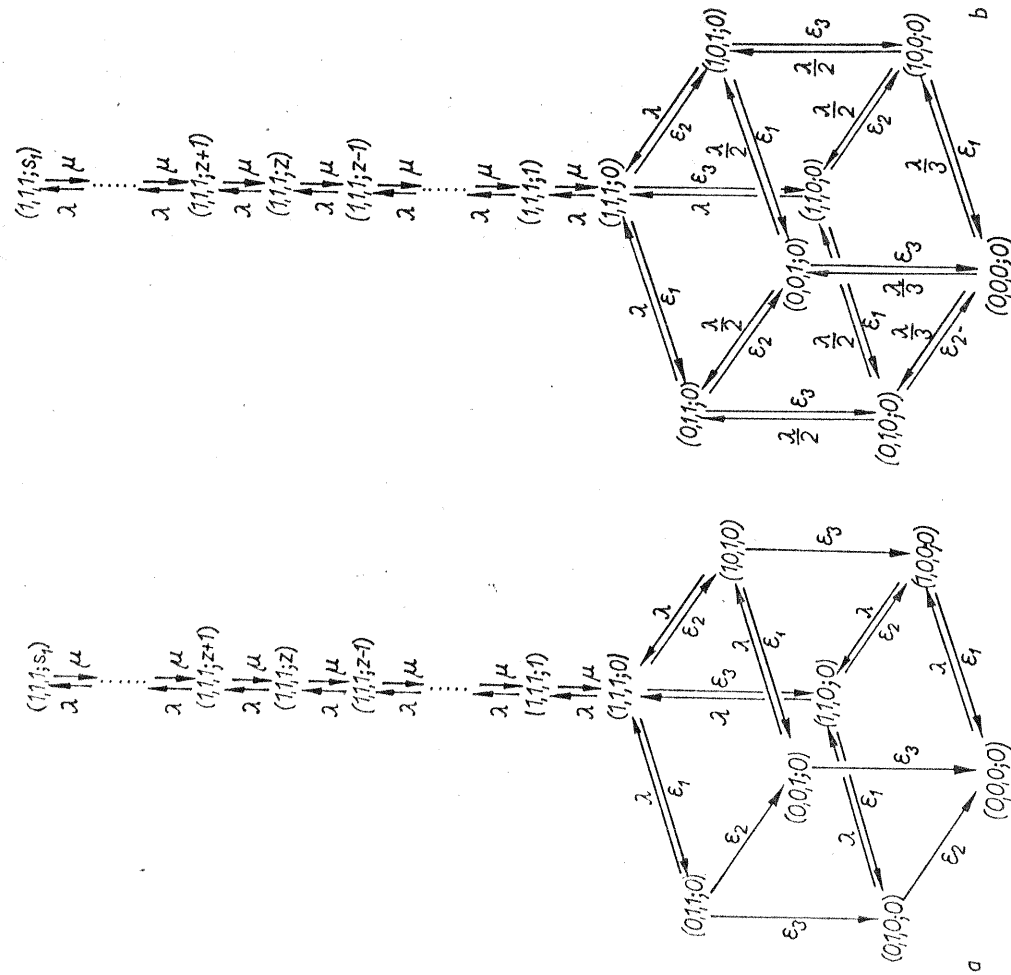$$\text{for } \prod_{\nu=1}^{n} x_\nu = 1.$$

Fig. 1. State space and transition coefficients for a 3-server combined delay and loss system a) sequential hunting of servers, b) random hunting of servers.

The steady state equations according to (5a, b) read as follows:

$$\left(\lambda + \sum_{\nu=1}^{n} x_\nu \varepsilon_\nu\right) \cdot p(x_1,\ldots,x_\nu,\ldots,x_n;0) = \lambda \cdot \sum_{\nu=1}^{n}\left(\prod_{x=1}^{\nu} x_x\right) p(\ldots,x_\nu-1,\ldots;0) +$$
$$+ \sum_{\nu=1}^{n}(1-x_\nu)\varepsilon_\nu \, p(\ldots,x_\nu+1,\ldots;0)$$
$$\text{for } \prod_{\nu=1}^{n} x_\nu = 0,$$

(7a)

The system of equations (9a), (7b, c) has the solutions

(10a) $p(x_1, \ldots, x_n; 0) = p(0, \ldots, 0; 0) \cdot \frac{(n-x)!}{n!} \cdot \prod_{v=1}^{n}\left(\frac{\lambda}{\varepsilon_v}\right)^{x_v}$, $x_v \in [0,1]$, $v = 1(1)n$,

(10b) $p(1, \ldots, 1; z) = p(0, \ldots, 0; 0) \cdot \frac{\varrho^z}{n!} \cdot \prod_{v=1}^{n}\left(\frac{\lambda}{\varepsilon_v}\right)$, $\qquad z = 0(1)s_1$,

(10c) $\dfrac{1}{p(0,\ldots,0;0)} = \sum_{x_1=0}^{1} \cdots \sum_{x_n=0}^{1} \dfrac{(n-x)!}{n!} \prod_{v=1}^{n}\left(\dfrac{\lambda}{\varepsilon_v}\right)^{x_v} + \dfrac{\varrho}{n!} \cdot \dfrac{1-\varrho^{s_1}}{1-\varrho} \cdot \prod_{v=1}^{n}\left(\dfrac{\lambda}{\varepsilon_v}\right)$.

The solution can be proved by insertion of (10a–c) into (9a) and (7b, c).

3.4. Identical servers

For identical termination rates

(11a) $\varepsilon_v = \varepsilon$, $\quad v = 1(1)n$,

(11b) $\mu = n\varepsilon$,

occupation patterns must not necessarily be distinguished. The state of the servers can be described by a single variable x, $x = 0(1)n$, representing the number of occupied servers. Then, the hunting discipline is meaningless. Combining the $\binom{n}{x}$ occupation patterns to a macrostate $(x; 0)$, $x = 0(1)n$, from (10a–c) follows:

(12a) $p(x; 0) = p(0; 0) \cdot \dfrac{A^x}{x!}$, $\qquad x = 0(1)n$,

(12b) $p(n; z) = p(0; 0) \cdot \dfrac{A^n}{n!} \cdot \varrho^z$, $\quad z = 0(1)s_1$,

(12c) $\dfrac{1}{p(0;0)} = \sum_{x=0}^{n} \dfrac{A^x}{x!} + \dfrac{A^n}{n!} \cdot \varrho \cdot \dfrac{1-\varrho^{s_1}}{1-\varrho}$,

where $A = \lambda/\varepsilon$ (traffic offered).

Finally, it should be noted that the probabilities of state are independent of the assumed queue disciplines.

4. CHARACTERISTIC VALUES

From the probabilities of state characteristic values for the quality of service can be derived which depend in general from the special queue disciplines. For abbreviation, let be $p(n; z) = p(1, \ldots, 1; z)$, $z = 0(1)s_1$.

4.1. Probability of waiting at arrival W

(13a) $W = \sum_{z=0}^{s_1-1} p(n; z) = p(n; 0) \cdot \dfrac{1-\varrho^{s_1}}{1-\varrho}$ for D1, D2, D3b,

(13b) $W = \sum_{z=0}^{s_1} p(n; z) = p(n; 0) \cdot \dfrac{1-\varrho^{s_1+1}}{1-\varrho}$ for D3a.

4.2. Probability of waiting successfully W^*

(14a) $W^* = W$ for D1, D2, D3b,

(14b) $W^* = \sum_{z=0}^{s_1-1} p(n; z) = p(n; 0) \cdot \dfrac{1-\varrho^{s_1}}{1-\varrho}$ for D3a.

4.3. Probability of waiting unsuccessfully W^{**}

(15a) $W^{**} = 0$ for D1, D2, D3b,

(15b) $W^{**} = p(n; s_1) = p(n; 0) \cdot \varrho^{s_1}$ for D3a.

4.4. Probability of loss at arrival B

(16a) $B = p(n; s_1) = p(n; 0) \cdot \varrho^{s_1}$ for D1, D2, D3b,

(16b) $B = 0$ for D3a.

4.5. Traffic carried Y

(17) $Y = \sum_{x_1=0}^{1} \cdots \sum_{x_n=0}^{1} (x_1 + \ldots + x_n) \cdot p(x_1, \ldots, x_n; 0) + n \cdot \sum_{z=1}^{s_1} p(n; z)$.

4.6. Mean queue length Ω

$$(18) \qquad \Omega = \sum_{z=0}^{s_1} z \cdot p(n; z) = p(n; 0) \cdot \varrho \cdot \left[\frac{1 - \varrho^{s_1}}{(1 - \varrho)^2} - \frac{s_1 \cdot \varrho^{s_1}}{1 - \varrho} \right].$$

4.7. Mean waiting time referred to all waiting calls t_W

$$(19a) \qquad t_W = \frac{\Omega}{\lambda \cdot W} = \frac{1}{\mu} \cdot \left[\frac{1}{1 - \varrho} - \frac{s_1 \cdot \varrho^{s_1}}{1 - \varrho^{s_1}} \right] \quad \text{for} \quad \text{D1, D2, D3b,}$$

$$(19b) \qquad t_W = \frac{\Omega}{\lambda \cdot W} = \frac{1}{\mu} \cdot \left[\frac{1}{1 - \varrho} \cdot \frac{1 - \varrho^{s_1}}{1 - \varrho^{s_1 + 1}} - \frac{s_1 \cdot \varrho^{s_1}}{1 - \varrho^{s_1 + 1}} \right] \quad \text{for} \quad \text{D3a.}$$

The mean waiting times of calls waiting successfully or in vain (D3a) will be given below by means of the 1st moments of the corresponding distribution functions of waiting time.

5. WAITING TIMES

The problem of waiting times is treated uniformly by introduction of a special waiting process which is constructed from the process of system states. This waiting process is described by backward-type equations for the conditional distribution functions of waiting time. By means of Laplace transformation we will treat the waiting time problem on the basis of the theory of eigenvalues. The influence of different queue disciplines on the waiting time is found by investigation of eigenvalues and distribution functions as well as by the higher moments.

A general theory of waiting times for Markovian queues has been developed by Syski [1], where the waiting time is considered as a "first passage time to a taboo set of absorbing states" in a "modified queueing process" which is constructed from the process of system states. This method will be applied for the treatment of waiting times under consideration of different queue disciplines.

5.1. Waiting process

For the determination of the waiting times we consider a "test call". The test call enters the system at an arbitrary instant ($t = 0$) of a blocking period and starts a special waiting process. This waiting process lasts as long as the test call is waiting

and finishes when the test call gets either service or becomes pushed out. Three types of waiting times can be distinguished:

T — waiting time = interval between the instant of arrival and either the instant at which service starts or the instant at which the call is being pushed out;

T^* — successful waiting time = interval between the instant of arrival and the instant at which service starts;

T^{**} — unsuccessful waiting time = interval between the instant of arrival and the instant at which the call is being pushed out.

Let $\zeta(t)$ be a random variable representing a pattern of all those calls in the system at time t which may influence the waiting time of the considered test call with respect to the underlying queue discipline (the test call is being excluded). The waiting process $\{\zeta(t), t \geq 0\}$ is constructed from the process of system states $\{\xi(t), t \geq 0\}$ by neglecting those transitions which cause no effect on the waiting time of the test call. The $\zeta(t)$-process has also the Markov property.

By H a taboo set of absorbing states for the test call will be understood. Entrance into H means termination of the $\zeta(t)$-process. H consists of all those states where the test call gets service, and a set of auxiliary states which characterize that the test call is being pushed out. By this definition the waiting time of a test call can be described as the first entrance time to the taboo set H [1].

5.2. Distribution function of waiting times

The $\zeta(t)$-process is described by conditional transition probabilities $_HP(t; j \mid i)$ which stand for the transition of the process from state i to j within time t under exclusion of the absorbing states. A conditional (complementary) distribution function of waiting time is defined by

$$(20) \qquad w(t \mid i) = \text{Prob}\{T > t \mid \zeta(0) = i\} = \sum_{j \notin H} {_HP(t; j \mid i)}, \quad i \notin H.$$

Because of the Markov property of the $\zeta(t)$-process, the conditional transition probabilities, and because of eq. (20) also the conditional distribution functions of waiting time, obey the Kolmogorov-backward-equation [1]:

$$(21) \qquad w'(t \mid i) = -q(i) \, w(t \mid i) + \sum_{\substack{j \neq i \\ j \notin H}} q(i, j) \, w(t \mid j), \quad i \notin H,$$

where $q(i, j)$ and $q(i)$ are the transition coefficients for the $\zeta(t)$-process.

The initial conditions $w(0|i) = 1$, $i \notin H$, satisfy the linear ... tem of equations (21) for $t = 0+$, where

(22a) $\qquad -w'(t|i)|_{t=0+} = \varepsilon(i)$, $i \notin H$.

$\varepsilon(i)$ are the transition coefficients for termination of the $\zeta(t)$-process from state i. Between $q(i)$, $q(i,j)$, and $\varepsilon(i)$ the following relation holds:

(22b) $\qquad q(i) = \sum_{\substack{j \neq i \\ j \notin H}} q(i,j) + \varepsilon(i)$, $i \notin H$.

REMARK. (20), (21), and (22a) hold also, if only successful or unsuccessful waiting calls are considered. Then, T, $w(t|i)$, and $\varepsilon(i)$ have to be substituted by T^* or T^{**}, $w^*(t|i)$ or $w^{**}(t|i)$, and $\varepsilon^*(i)$ or $\varepsilon^{**}(i)$, respectively, while $q(i,j)$ and $q(i)$ remain unchanged. The conditional probabilities for success or failure, $w^*(0|i)$ or $w^{**}(0|i)$, can be calculated from (21) for $t = 0+$. The following relations hold:

(23a) $\qquad w^*(t|i) + w^{**}(t|i) = w(t|i)$, $i \notin H$.

(23b) $\qquad \varepsilon^*(i) + \varepsilon^{**}(i) = \varepsilon(i)$, $i \notin H$.

The absolute distribution function of waiting time referred to a call which met an arbitrary state at its arrival is given by

(24) $\qquad W(>t) = \mathrm{Prob}\{T > t\} = \sum_{i \notin H} P(i)\, w(t|i)$,

where

$$P(i) = \mathrm{Prob}\{\zeta(0) = i\}, \quad i \notin H,$$

denotes the initial distribution of states. This initial distribution can be calculated from the stationary probabilities of state and depends on the queue discipline. Again, (24) holds also for calls waiting successfully or in vain, if T and $w(t|i)$ are substituted by T^* or T^{**} and $w^*(t|i)$ or $w^{**}(t|i)$, respectively.

5.3. Laplace transform and eigenvalue problem

Be

(25) $\qquad W(s|i) = \int_{t=0}^{\infty} w(t|i)\, e^{-st}\, dt$

the Laplace transform of $w(t|i)$, where $s = \sigma + j\omega$ denotes the complex variable. Applying (25) to (21) we get

(26) $\qquad [q(i) + s] \cdot W(s|i) - \sum_{\substack{j \neq i \\ j \notin H}} q(i,j)\, W(s|j) = w(0|i)$, $i \notin H$.

The coefficients of system (26) can be written in matrix notation as $A + sI$, where $A = (-q(i,j))$, $-q(i,i) = q(i)$, denotes the matrix of transition coefficients, and I the unit matrix. The order of both matrices be k.

The solution of (26) yields rational functions of s, where the denominator polynomial is identical to $\det(A + sI)$. By partial fraction expansion we obtain

(27) $\qquad W(s|i) = \sum_{\nu=1}^{m} \sum_{\varkappa=1}^{m_\nu} \frac{a_{i\nu}^{(\varkappa)}}{(s - e_\nu)^\varkappa}$, $i \notin H$.

In (27) e_ν, $\nu = 1(1)m$, are the distinct zeros of the denominator polynomial; m_ν denotes the multiplicity of zero e_ν, $\nu = 1(1)m$, $\varkappa = 1(1)m_\nu$, $i \notin H$, are the residues of the partial fraction expansion. Between k, m, and m_ν the relation $\sum_{\nu=1}^{m} m_\nu = k$ holds.

Inverse Laplace transformation of (27) yields

(28) $\qquad w(t|i) = \sum_{\nu=1}^{m} e^{e_\nu t} \cdot \sum_{\varkappa=1}^{m_\nu} a_{i\nu}^{(\varkappa)} \frac{t^{\varkappa-1}}{(\varkappa-1)!}$, $i \notin H$.

With (24) and (28) the absolute distribution function of waiting time is known. The main difficulty arises in finding the roots (eigenvalues) of

(29) $\qquad \det(A + sI) = 0$.

Hence, the investigation of the distribution function of waiting time starts with the investigation of eigenvalues.

5.4. Moments of the distribution density function of waiting time

The K-th conditional moment of the conditional distribution density function of waiting time $h(t|i) = -w'(t|i)$ is defined by

(30) $\qquad M_K(i) = \int_{t=0}^{\infty} t^K h(t|i)\, dt$.

Let $H(s|i)$ be the Laplace transform of $h(t|i)$. By the correspondence

(31) $\qquad (-1)^K t^K h(t|i) \overset{\mathscr{L}}{\rightarrow} \frac{d^K}{ds^K} H(s|i)$

follows

(32) $\qquad M_K(i) = \lim_{s \to 0} (-1)^K \frac{d^K}{ds^K} H(s|i)$.

Differentiating eq. (21) with respect to t and application of ⌢ place transformation the following equation will be obtained:

$$(33) \qquad [q(i) + s] \cdot H(s|i) - \sum_{\substack{j \neq i \\ j \notin H}} q(i,j)\, H(s|j) = \varepsilon(i), \quad i \notin H.$$

Successive differentiation of (33) with respect to s and letting $s \to 0$ leads to the following recursion formulas for the conditional moments:

$$(34) \qquad q(i)\, M_K(i) - \sum_{\substack{j \neq i \\ j \notin H}} q(i,j)\, M_K(j) = K \cdot M_{K-1}(i),$$

where

$$M_0(i) = w(0|i), \quad i \notin H.$$

In general, for $K = 1, 2, \ldots$ a system of linear equations is to solve. In some special cases, however, the conditional moments can be given explicitly or by means of recursion.

The total moments of the distribution density function of waiting time, M_K, are obtained from (24) analogously:

$$(35) \qquad M_K = \sum_{i \notin H} P(i)\, M_K(i), \quad K = 1, 2, \ldots$$

REMARK. The first conditional moments $M_1(i)$ are equal to the conditional mean waiting times $t_W(i)$, $i \notin H$. Between the first total moment, M_1, and the mean waiting time referred to all waiting calls, t_W, relation (36) holds:

$$(36) \qquad M_1 = W t_W.$$

t_W has already been derived by the mean queue length, c.f. section 4.7. The conditional mean waiting times can only be calculated from a linear system of equations according to (34). The total mean waiting times for calls waiting successfully and in vain, t_W^* and t_W^{**}, cannot be determined from the mean queue length; they have to be calculated from the conditional mean waiting times of calls waiting successfully and in vain, $t_W^*(i)$ and $t_W^{**}(i)$, $i \notin H$, respectively.

6. SPECIAL QUEUE DISCIPLINES

The general theory will be applied to the four different queue disciplines D1, D2, D3a, and D3b. By definition, $\zeta(t)$ consists of all those calls being in service and waiting which may have an influence on the waiting time of a test call. The n calls being in service have in any case an influence on the waiting time of a test call.

Among the waiting calls, only those have to be considered which arrived prior to the test call (D1), after the test call (D3a), or both (D2, D3b).

6.1. First-come, first-served service (D1)

Be $\zeta(t) = i$, $i = 0(1)s_1 - 1$, the random variable of the waiting process, where i denotes the number of calls waiting in front of the test call with initial distribution

$$(37) \qquad P(i) = \text{Prob}\{\zeta(0) = i\} = p(n;i), \quad i = 0(1)s_1 - 1.$$

Fig. 2a illustrates the states and transitions of the $\zeta(t)$-process. With transition

Fig. 2. States and transitions of the waiting process
a) first-come, first-served service (D1), b) random service (D2).

coefficients from Fig. 2a the system of differential equations for the conditional distribution functions of waiting time, $w(t|i)$, can be stated according to (21) and (22b)

$$(38a) \qquad w'(t|0) = -\mu w(t|0),$$

$$(38b) \qquad w'(t|i) = -\mu w(t|i) + \mu w(t|i-1), \quad i = 1(1)s_1 - 1,$$

with initial conditions $w(0|i) = 1$, $i = 0(1)s_1 - 1$.

The matrix A of (38a, b) is bidiagonal

$$(39)\qquad A = \begin{pmatrix} \mu & & & \\ -\mu & \mu & & \\ & \ddots & \ddots & \\ & & -\mu & \mu \\ & & & -\mu & \mu \end{pmatrix}.$$

Successive expansion of $\det(A + sI)$ leads to

$$(40)\qquad \det(A + sI) = (s + \mu)^{s_1},$$

i.e. there exists $m = 1$ eigenvalue with multiplicity $m_1 = s_1$:

$$(41)\qquad e_1 = -\mu.$$

The Laplace transforms $W(s|i)$, $i = 0(1)s_1 - 1$, can be calculated by recursion

$$(42)\qquad W(s|i) = \sum_{\nu=1}^{i+1} \frac{\mu^{\nu-1}}{(s+\mu)^\nu}, \quad i = 0(1)s_1 - 1.$$

The inverse Laplace transformation of eq. (42) yields

$$(43)\qquad w(t|i) = \sum_{\nu=0}^{i} \frac{(\mu t)^\nu}{\nu!} \cdot e^{-\mu t}, \quad i = 0(1)s_1 - 1,$$

what is, obviously, an Erlang-$(i+1)$-distribution. The total distribution function of waiting times follows from (24), (37), and (43) after some rearrangements:

$$(44)\qquad W(>t) = \frac{W}{1 - \varrho^{s_1}} \left[\sum_{\nu=0}^{s_1-1} \frac{(\lambda t)^\nu}{\nu!} - \varrho^{s_1} \cdot \sum_{\nu=0}^{s_1-1} \frac{(\mu t)^\nu}{\nu!} \right] \cdot e^{-\mu t}.$$

The result (44) has already been obtained in 1956 by H. Störmer [3] by a more direct method.

The equations for the conditional moments follow from (38a, b)

$$(45a)\qquad \mu M_K(0) = K M_{K-1}(0),$$

$$(45b)\qquad \mu M_K(i) - \mu M_K(i-1) = K M_{K-1}(i), \quad i = 1(1)s_1 - 1.$$

The solution of (45a, b) is obtained by recursion

$$(46)\qquad M_K(i) = \frac{1}{\mu^K} \cdot \frac{(i+K)!}{i!}.$$

The total moments follow from (35), (37), and (46)

$$(47)\qquad M_K = p(n; 0) \cdot \frac{1}{\mu^K} \cdot \sum_{i=0}^{s_1-1} \frac{(i+K)!}{i!} \cdot \varrho^i, \quad K = 1, 2, \ldots$$

6.2. Random order of service (D2)

For D2 all waiting calls compete for a server becoming idle. Be $\zeta(t) = i$, $i = 0(1)s_1 - 1$, the random variable with initial distribution eq. (37), where i denotes the number of waiting calls competing with the test call for service.

According to the illustration of the waiting process Fig. 2b, the equations read

$$(48a)\qquad w'(t|i) = -(\lambda + \mu) \cdot w(t|i) + \lambda \cdot w(t|i+1) + \frac{i}{i+1} \cdot \mu \cdot w(t|i-1),$$
$$i = 0(1)s_1 - 2,$$

$$(48b)\qquad w'(t|s_1 - 1) = -\mu \cdot w(t|s_1 - 1) + \frac{s_1-1}{s_1} \mu \cdot w(t|s_1 - 2), \quad i = 0(1)s_1 - 1.$$

with initial conditions $w(0|i) = 1$, $i = 0(1)s_1 - 1$.

Note, when there are i calls competing with the test call for a free server, the test call will not be selected with probability $i/(i+1)$.

The matrix A is of the unsymmetrical tridiagonal type [4]:

$$(49)\qquad A = \begin{pmatrix} (\lambda+\mu) & -\lambda & & & & \\ -\frac{1}{2}\mu & (\lambda+\mu) & -\lambda & & & \\ & \ddots & \ddots & \ddots & & \\ & & -\frac{i-1}{i}\mu & (\lambda+\mu) & -\lambda & \\ & & & \ddots & \ddots & \ddots \\ & & & & -\frac{s_1-2}{s_1-1}\mu & (\lambda+\mu) & -\lambda \\ & & & & & -\frac{s_1-1}{s_1}\mu & \mu \end{pmatrix}$$

THEOREM 1. Matrix eq. (49) has the following three properties:

i) the eigenvalues are negative-real,
ii) the eigenvalues are distinct,

iii) *the eigenvalues are placed within the interval*

$$\left[-\max\left(2\lambda + \frac{2s_1-3}{s_1-1}\mu,\ \frac{2s_1-1}{s_1}\mu\right),\ -\frac{\mu}{s_1}\right],\quad s_1 \geqq 2.$$

(For the special cases $s_1 \leqq 4$ the eigenvalues can be given explicitly.)

Proof i. The unsymmetrical tridiagonal matrix (49) belongs to the class of symmetrizable matrices. Symmetrizable matrices can be transformed to symmetrical form by a nonsingular linear transformation. They can be written as product $A = BC$, where both B and C are symmetrical and, moreover, one of B or C has to be positive definite. If both B and C are positive definite, all the eigenvalues of A are negative-real [5].

Matrix A can be decomposed by the following matrices B and C:

$$(50)\quad B = \begin{pmatrix} 1 & & \cdots \\ & \frac{1}{i\varrho^{i-1}} & \cdots \\ \cdots & & \frac{1}{s_1\varrho^{s_1-1}} \end{pmatrix},\quad C = \begin{pmatrix} (\lambda+\mu) & -\lambda & & \cdots \\ -\lambda & (\lambda+\mu)2\varrho & -2\lambda\varrho & \cdots \\ \cdots & -(i-1)\lambda\varrho^{i-2} & (\lambda+\mu)i\varrho^{i-1} & -i\lambda\varrho^{i-1} & \cdots \\ \cdots & & -(s_1-1)\lambda\varrho^{s_1-2} & \mu s_1\varrho^{s_1-1} \end{pmatrix}$$

Both B and C are symmetrical and positive definite. The latter can be shown by positive principal subdeterminants.

Proof ii. Consider the sequence $D_i(s)$, $i = 0(1)s_1$, of principal subdeterminants of $(A + sI)$. By recursive expansion follows

$$(51a)\quad D_i(s) = (s+\lambda+\mu)D_{i-1}(s) - \frac{i-1}{i}\lambda\mu D_{i-2}(s),\quad i = 1(1)s_1 - 1,$$

$$(51b)\quad D_{s_1}(s) = (s+\mu)D_{s_1-1}(s) - \frac{s_1-1}{s_1}\lambda\mu D_{s_1-2}(s),$$

where $D_0 = 1$, $D_{-1} = 0$. Because of $D_i(0) > 0$, $i = 0(1)s_1$, and $\lim_{\sigma\to-\infty} D_i(\sigma) = (-1)^i\infty$, $i = 1(1)s_1$, it can be shown recursively from $i = 1$ up to s_1 that the roots of $D_{i-1}(s)$ interlace the roots of $D_i(s)$. Hence, the roots of $D_{s_1}(s)$, i.e. the eigenvalues of A, are negative-real and distinct.

Proof iii. According to a theorem of Gershgorin [5], the eigenvalues of $A = (-q(i,j))$ are located in the intersection of the two regions G and G' which are built from the circles K_i and K_i' with centre $-q(i)$ and radius $r_i = \sum_{j\neq i} q(i,j)$ or $r_i' = \sum_{j\neq i} q(j,i)$, $i = 0(1)s_1 - 1$, respectively. Application of this theorem on matrix A

$= \sum_{j\neq i} q(j,i)$, $i = 0(1)s_1 - 1$, yields the mentioned interval.

Because of the properties i) and ii), (24) and (37), the total distribution function of waiting time is given by

$$(52)\quad W(>t) = \sum_{i=0}^{s_1-1} p(n;i) \cdot \sum_{v=1}^{s_1} a_{iv} \cdot e^{e_v t}.$$

The eigenvalues have to be calculated numerically.

The conditional moments obey to the following linear system of equations:

$$(53a)\quad (\lambda + \mu) M_K(i) - \lambda M_K(i+1) - \frac{i}{i+1}\mu M_K(i-1) = K M_{K-1}(i),$$
$$i = 0(1)s_1 - 2.$$

$$(53b)\quad \mu M_K(s_1 - 1) - \frac{s_1-1}{s_1}\mu M_K(s_1 - 2) = K M_{K-1}(s_1 - 1).$$

Generally, this system has to be solved as a whole for each K. However, the first conditional moments, which allow the calculation of the first and second total moment, can be determined by the following recursive algorithm [4]:

$$(54)\quad M_1(i) = M_1(0) \cdot P_i(\xi) - \frac{1}{\lambda} \cdot Q_i(\xi),\quad i = 1(1)s_1 - 1,$$

where

$$P_i(\xi) = (1 + \xi) \cdot P_{i-1}(\xi) - \frac{i-1}{i} \cdot \xi P_{i-2}(\xi),\qquad P_0(\xi) = 1,\qquad i = 1(1)s_1 - 1,$$

$$Q_i(\xi) = (1 + \xi) \cdot Q_{i-1}(\xi) - \frac{i-1}{i} \cdot \xi Q_{i-2}(\xi) + 1,\quad Q_0(\xi) = 0,$$

$$M_1(0) = \frac{1}{\mu} \cdot \frac{1}{P_{s_1-1}(\xi)} \cdot \frac{1+\xi \cdot \{Q_{s_1-1}(\xi) - [(s_1 - 1)/s_1] \cdot Q_{s_1-2}(\xi)\}}{P_{s_1-1}(\xi) - [(s_1 - 1)/s_1] \cdot P_{s_1-2}(\xi)},\quad \xi = \frac{1}{\varrho}.$$

According to a procedure given by M. Segal for the finite source queueing model [6], the K-th total moment can be given in terms of the conditional $(K - 1)$st moments.

Multiplication of (53a, b) by $(i + 1) \cdot p(n; i)$ and summation over i yields

$$(55) \quad M_K = \frac{1}{\mu} \cdot \sum_{i=0}^{s_1-1} p(n; i)(i + 1) K M_{K-1}(i), \quad K = 1, 2, \ldots$$

For $K = 1$ the same result as in case of D1, eq. (47), is obtained. Clearly, the mean waiting times are not influenced by the queue discipline. For $K = 2$ from (55) follows that the second total moment M_2 can be calculated from the first conditional moments. Furthermore, it can be shown that for D2 the second total moment M_2 exceeds the corresponding one for D1 [4].

6.3. Last-come, first-served service with push-out priority (D3a)

An arriving call occupies principally the first waiting place (when all the servers are busy) and displaces all the other waiting calls. If the arriving call finds a fully occupied queue, the waiting call with the greatest waiting time will be lost (pushed out).

Be $\zeta(t) = i$, $i = 0(1) s_1 - 1$, the random variable of the waiting process, where i denotes the number of waiting calls in front of the test call. The waiting process for the total waiting time starts in any case with $\zeta(0) = 0$. Hence, the initial distribution is

$$(56) \quad P(i) = \begin{cases} W & \text{for } i = 0, \\ 0 & \text{for } i > 0. \end{cases}$$

For $\zeta(0) > 0$ only a waiting process of a previously waiting call can develop (partial waiting time). The waiting process finishes when the test call is either served or pushed out.

Fig. 3a illustrates the waiting process which is described by the following equations:

$$(57a) \quad w'(t \mid 0) = -(\lambda + \mu) \cdot w(t \mid 0) + \lambda \cdot w(t \mid 1),$$

$$(57b) \quad w'(t \mid i) = -(\lambda + \mu) \cdot w(t \mid i) + \lambda \cdot w(t \mid i + 1) + \mu \cdot w(t \mid i - 1),$$
$$i = 1(1) s_1 - 2,$$

$$(57c) \quad w'(t \mid s_1 - 1) = -(\lambda + \mu) \cdot w(t \mid s_1 - 1) \qquad + \mu \cdot w(t \mid s_1 - 2).$$

REMARK. (57a—c) hold also for calls waiting successfully or in vain, if $w(t \mid i)$ is substituted by $w^*(t \mid i)$ or $w^{**}(t \mid i)$, $i = 0(1) s_1 - 1$, respectively. The difference lies in the initial conditions which can be calculated from (57a—c) regarding

$$(58a) \quad \varepsilon^*(i) = \begin{cases} \mu & \text{for } i = 0, \\ 0 & \text{for } i = 1(1) s_1 - 1, \end{cases}$$

$$(58b) \quad \varepsilon^{**}(i) = \begin{cases} 0 & \text{for } i = 0(1) s_1 - 2, \\ \lambda & \text{for } i = s_1 - 1. \end{cases}$$

The results for the initial conditions are

$$(59a) \quad w^*(0 \mid i) = \frac{1 - \varrho^{s_1-i}}{1 - \varrho^{s_1+1}},$$

$$(59b) \quad w^{**}(0 \mid i) = \frac{\varrho^{s_1-i} - \varrho^{s_1+1}}{1 - \varrho^{s_1+1}}, \quad i = 0(1) s_1 - 1.$$
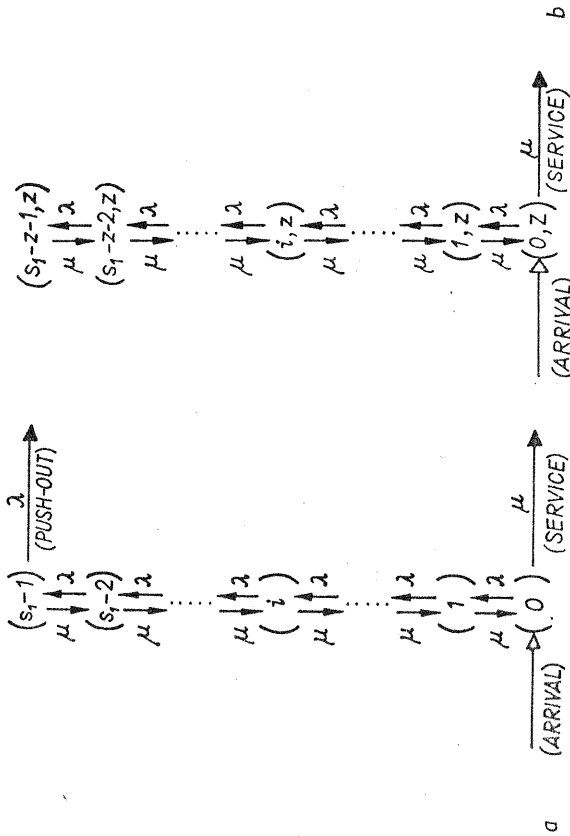

Fig. 3. States and transitions of the waiting process for last-come, first-served service (D3) a) with push-out priority, b) without push-out priority.

The matrix A has a special form of the tridiagonal type [4]:

$$(60) \quad A = \begin{pmatrix} (\lambda + \mu) & -\lambda & & & & \\ -\mu & (\lambda + \mu) & -\lambda & & & \\ & -\mu & (\lambda + \mu) & -\lambda & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\mu & (\lambda + \mu) & -\lambda \\ & & & & -\mu & (\lambda + \mu) \end{pmatrix}$$

THEOREM 2. *The matrix (60) has s_1 distinct negative-real eigenvalues*

$$(61) \quad e_\nu = 2\sqrt{(\lambda\mu)} \cdot \cos\frac{\nu\pi}{s_1 + 1} - (\lambda + \mu), \quad \nu = 1(1) s_1.$$

Proof. By the nonsingular linear transformation

$$(62) \quad B = T^{-1} A T,$$

where $T = (\sqrt{(\xi)^{i-1}} \cdot \delta_{ij})$, $i = 1(1)s_1$, $\xi = 1/\varrho$, (δ_{ij} Kronecker-symbol) follows

(63)
$$B = \sqrt{\lambda\mu} \begin{pmatrix} \dfrac{\lambda+\mu}{\sqrt{\lambda\mu}} & -1 & & \cdots \\ -1 & \dfrac{\lambda+\mu}{\sqrt{\lambda\mu}} & -1 & \cdots \\ \cdots & -1 & \dfrac{\lambda+\mu}{\sqrt{\lambda\mu}} & -1 \\ \cdots & & -1 & \dfrac{\lambda+\mu}{\sqrt{\lambda\mu}} \end{pmatrix}.$$

The eigenvalues are invariant with respect to a nonsingular linear transformation (62). Matrix B according to (63) is the well-known difference matrix with the eigenvalues (61) [5].

THEOREM 3. *The total distribution function of waiting time is given by*

(64)
$$W(>t) = W \cdot \sum_{v=1}^{s_1} \frac{\sum_{i=1}^{s_1} w(0\,|\,i-1)\,\lambda^{i-1} \cdot D_{s_1-i}(e_v)}{\prod_{i \neq v}(e_v - e_i)} \cdot e^{e_v t},$$

$v = 1(1)s_1$, $k = 0(1)s_1 - 1$.

where

$$D_k(e_v) = \sqrt{\lambda\mu}^k \cdot \frac{\sin[(k+1)\,v\pi/(s_1+1)]}{\sin[v\pi/(s_1+1)]}, \quad v = 1(1)s_1, \quad k = 0(1)s_1 - 1.$$

Proof. Next, from (24) and (56) follows

(65)
$$W(>t) = W \cdot w(t\,|\,0).$$

From the transformed system (57a–c) only $W(s\,|\,0)$ has to be calculated. Be

(66)
$$W(s\,|\,0) = \frac{N(s\,|\,0)}{D_{s_1}(s)},$$

where

(67)
$$D_{s_1}(s) = \det(A + sI) = \prod_{i=1}^{s_1}(s - e_i).$$

$N(s\,|\,0)$ is obtained from $\det(A + sI)$ by substitution of the first column vector by the vector of initial conditions. Thus,

(68)
$$N(s\,|\,0) = \sum_{i=1}^{s_1} w(0\,|\,i-1)\,\lambda^{i-1}\, D_{s_1-i}(s),$$

where $D_k(s)$ means the k-th principal subdeterminant of $(A + sI)$. From (66), (67), and (68) follows by partial fraction expansion

(69)
$$W(s\,|\,0) = \sum_{v=1}^{s_1} \frac{a_{0v}}{s - e_v},$$

where

$$a_{0v} = \frac{\sum_{i=1}^{s_1} w(0\,|\,i-1)\,\lambda^{i-1}\, D_{s_1-i}(e_v)}{\prod_{i \neq v}(e_v - e_i)}, \quad v = 1(1)s_1.$$

In (69) the principal subdeterminants $D_k(e_v)$, $v = 1(1)s_1$, $k = 0(1)s_1 - 1$, have still to be determined. Because of $\det(A + sI) = \det T^{-1}(A + sI)T$, where $T = (\sqrt{(\xi)^{i-1}} \cdot \delta_{ij})$, and the special eigenvalues according to (61) we obtain

(70)
$$D_k(e_v) = \sqrt{\lambda\mu}^k \cdot \begin{vmatrix} 2\cos\dfrac{v\pi}{s_1+1} & -1 & \cdots \\ -1 & 2\cos\dfrac{v\pi}{s_1+1} & -1 \\ \cdots & -1 & 2\cos\dfrac{v\pi}{s_1+1} \end{vmatrix},$$

$v = 1(1)s_1$, $k = 0(1)s_1 - 1$.

Substituting

$$2\cos\frac{v\pi}{s_1+1} = e^{ju_v} + e^{-ju_v},$$

where

$$u_v = \frac{v\pi}{s_1+1}, \quad v = 1(1)s_1,$$

the determinant (70) can be developed

(71)
$$D_k(e_v) = \sqrt{\lambda\mu}^k \cdot \sum_{x=0}^{k}(e^{ju_v})^{k-2x} = \sqrt{\lambda\mu}^k \cdot \frac{\sin[(k+1)\,v\pi/(s_1+1)]}{\sin[v\pi/(s_1+1)]},$$

$v = 1(1)s_1$, $k = 0(1)s_1 - 1$.

Inverse Laplace transformation of (69) yields $w(t\,|\,0)$, and with (65), finally, the total distribution function of waiting time (64).

REMARK. The distribution functions of waiting time for successful waiting calls and calls waiting in vain, $W^*(>t)$ and $W^{**}(>t)$, can also be derived from theorem 3. For this, in (64) the initial conditions $w(0|i)$, $i = 0(1)s_1 - 1$, have to be substituted by $w^*(0|i)$ and $w^{**}(0|i)$, $i = 0(1)s_1 - 1$, according to (59a, b), respectively.

For the conditional moments the following system of equations holds:

(72a) $(\lambda + \mu) M_K(0) \quad -\lambda M_K(1) \qquad = K M_{K-1}(0)$,

(72b) $(\lambda + \mu) M_K(i) \quad -\lambda M_K(i+1) - \mu M_K(i-1) = K M_{K-1}(i)$,
$$i = 1(1)s_1 - 2,$$

(72c) $(\lambda + \mu) M_K(s_1 - 1) \qquad -\mu M_K(s_1 - 2) = K M_{K-1}(s_1 - 1)$.

For recursive solution of this system two different methods are possible.

First, from the known distribution function $w(t|0)$ the conditional moments $M_K(0)$ can be derived directly by integration:

(73)
$$M_K(0) = \sum_{v=1}^{s_1} a_{0v} \cdot \frac{K!}{(-e_v)^K}, \quad K = 1, 2, \ldots$$

With the aid of $M_K(0)$ from (72a–c) all conditional moments can be calculated recursively by

(74)
$$M_K(i) = M_K(0) \cdot \sum_{v=1}^{i} \zeta^v - \frac{K}{\lambda} \cdot \left[\sum_{v=0}^{i-1} M_{K-1}(v) \sum_{x=0}^{i-v-1} \zeta^x \right],$$
$$i = 1(1)s_1 - 1, \quad K = 1, 2, \ldots,$$

where $\zeta = 1/\varrho$. For this first recursion algorithm the knowledge of eigenvalues is presumed.

Second, with the aid of recursion procedure (74) the following formula for the conditional moments $M_K(0)$ can be derived:

(75)
$$M_K(0) = \frac{K}{\mu} \cdot \frac{1}{1 - \varrho^{s_1+1}} \cdot \sum_{v=0}^{s_1-1} M_{K-1}(v)(\varrho^v - \varrho^{s_1}), \quad K = 1, 2, \ldots$$

Eq. (74) and (75) allow the recursive calculation of all conditional moments without the knowledge of eigenvalues.

The total moments are derived from (65)

(76)
$$M_K = W M_K(0), \quad K = 1, 2, \ldots$$

REMARK. The above procedures for the calculation of moments hold also when calls are considered which wait successfully or in vain. For this, the values $M_0(i) =$
$= w(0|i) = 1$, $i = 0(1)s_1 - 1$, have to be substituted by $M_0^*(i) = w^*(0|i)$ and $M_0^{**}(i) = w^{**}(0|i)$, $i = 0(1)s_1 - 1$, respectively.

The mean waiting time referred to all waiting calls, t_W, referred to all successful waiting calls, t_W^*, and referred to all calls waiting in vain, t_W^{**}, follow from the corresponding first total moments:

(77a) $t_W = M_1/W$,

(77b) $t_W^* = M_1^*/W^*$,

(77c) $t_W^{**} = M_1^{**}/W^{**}$.

t_W has already been calculated by the mean queue length Ω, (19b).

6.4. Last-come, first-served service without push-out priority (D3b)

An arriving call occupies the first waiting place when it finds all the servers busy and if there is at least one waiting place free. If the arriving call meets the system fully occupied it will be rejected at once. For this discipline, besides the n calls being in service both, waiting calls which arrived prior to the test call, as well as calls which may have been accepted after the test call have an influence on the waiting time of the test call.

Be $\zeta(t) = (i, z)$, $i = 0(1)s_1 - z - 1$, $z = 0(1)s_1 - 1$, the random variable of the waiting process, where i denotes the number of waiting calls in front of the test call; z indicates the number of waiting calls which the test call has found at its arrival. For each $z = 0(1)s_1 - 1$ a specific waiting process develops: for each waiting process z remains constant. Hence, the initial distribution is

(78)
$$P(i, z) = \begin{cases} p(n; z) & \text{for } i = 0, \\ 0 & \text{for } i > 0, \end{cases} \qquad z = 0(1)s_1 - 1.$$

In Fig. 3b the waiting process is illustrated for a general z, for which the following equations hold:

(79a) $w'(t|0, z) = -(\lambda + \mu) \cdot w(t|0, z) \quad + \lambda \cdot w(t|1, z)$.

(79b) $w'(t|i, z) = -(\lambda + \mu) \cdot w(t|i, z) \quad + \lambda \cdot w(t|i+1, z) +$
$$+ \mu \cdot w(t|i-1, z), \quad i = 1(1)s_1 - z - 2,$$

(79c) $w'(t|s_1 - z - 1, z) = -\mu \cdot w(t|s_1 - z - 1, z) + \mu \cdot w(t|s_1 - z - 2, z)$.

with initial conditions $w(0|i, z) = 1$, $i = 0(1)s_1 - z - 1$, $z = 0(1)s_1 - 1$.

Fig. 4. Distribution function of waiting time ($n = 1$, $s_1 = 12$).

(——— D1; −−−− D2; −·−·− D3a, calls waiting successfully (*); −··−··− D3a, calls waiting in vain (**); ······· D3b.)

The solutions for the first and second conditional moments are as follows:

$$(83) \quad M_1(i, z) = \frac{1}{\mu} \, \frac{1}{(1-\varrho)^2} \cdot [(i+1)(1-\varrho) + \varrho^{s_1-z-i} \cdot (\varrho^{i+1} - 1)].$$

$$(84a) \quad M_2(i, z) = M_2(0, z) \cdot \sum_{v=0}^{i} \xi^v - \frac{2}{\lambda} \cdot [\sum_{v=0}^{i-1} M_1(v, z) \cdot \sum_{x=0}^{v-1} \xi^x], \quad \xi = 1/\varrho,$$

where

$$(84b) \quad M_2(0, z) = \frac{2}{\mu^2} \cdot \frac{1}{(1-\varrho)^2} \cdot \left[\frac{1 - \varrho^{2(s_1-z)+1}}{1 - \varrho} - [2(s_1 - z) + 1] \cdot \varrho^{s_1-z} \right].$$

The matrix A is of the unsymmetrical tridiagonal type [4]:

$$(80) \quad A = \begin{pmatrix} (\lambda + \mu) & -\lambda & \cdots & & & \\ -\mu & (\lambda + \mu) & -\lambda & \cdots & & \\ \cdots & \cdots & \cdots & \cdots & & \\ & & -\mu & (\lambda + \mu) & -\lambda \\ & & & -\mu & \mu \end{pmatrix}.$$

THEOREM 4. *Matrix (80) has the following three properties:*

i) *the eigenvalues are negative-real,*

ii) *the eigenvalues are distinct,*

iii) *the eigenvalues are placed within the interval* $[-2(\lambda + \mu), 0]$, $s_1 \geqq 2$.

(For the special cases $s_1 \leqq 4$ the eigenvalues can be given explicitly.)

Proof i. Application of the nonsingular linear transformation (62), where $T = (\sqrt{(\xi)}^{i-1} \cdot \delta_{ij})$, $i = 1(1) s_1 - z$, yields a matrix with positive principal subdeterminants, i.e. a positive definite matrix.

Proof ii. The sequence $D_i(s)$, $i = 0(1) s_1 - z$, of principal subdeterminants of $(A + sI)$ is considered in a recursive representation. The proof follows then the same line as in section 6.2.

Proof iii. By application of Gershgorin's theorem [5] the interval $[-2(\lambda + \mu), 0]$ is found. Since matrix A is nonsingular (c.f. proof i) the eigenvalue zero cannot be assumed.

According to properties i) and ii), (24) and (78), the total distribution function of waiting time is given by

$$(81) \quad W(>t) = \sum_{z=0}^{s_1-1} p(n; z) \, w(t \mid 0, z) = \sum_{z=0}^{s_1-1} p(n; z) \cdot \sum_{v=1}^{s_1-z} a_{zv} \, e^{e_{zv}t}.$$

The eigenvalues e_{zv}, $v = 1(1) s_1 - z$, $z = 0(1) s_1 - 1$, have to be calculated numerically from the different system for $z = 0(1) s_1 - 1$.

The conditional moments obey to the following equations:

$$(82a) \quad (\lambda + \mu) M_K(0, z) - \lambda M_K(1, z) = K M_{K-1}(0, z),$$

$$(82b) \quad (\lambda + \mu) M_K(i, z) - \lambda M_K(i + 1, z) - \mu M_K(i - 1, z) = K M_{K-1}(i, z), \quad i = 1(1) s_1 - z - 2,$$

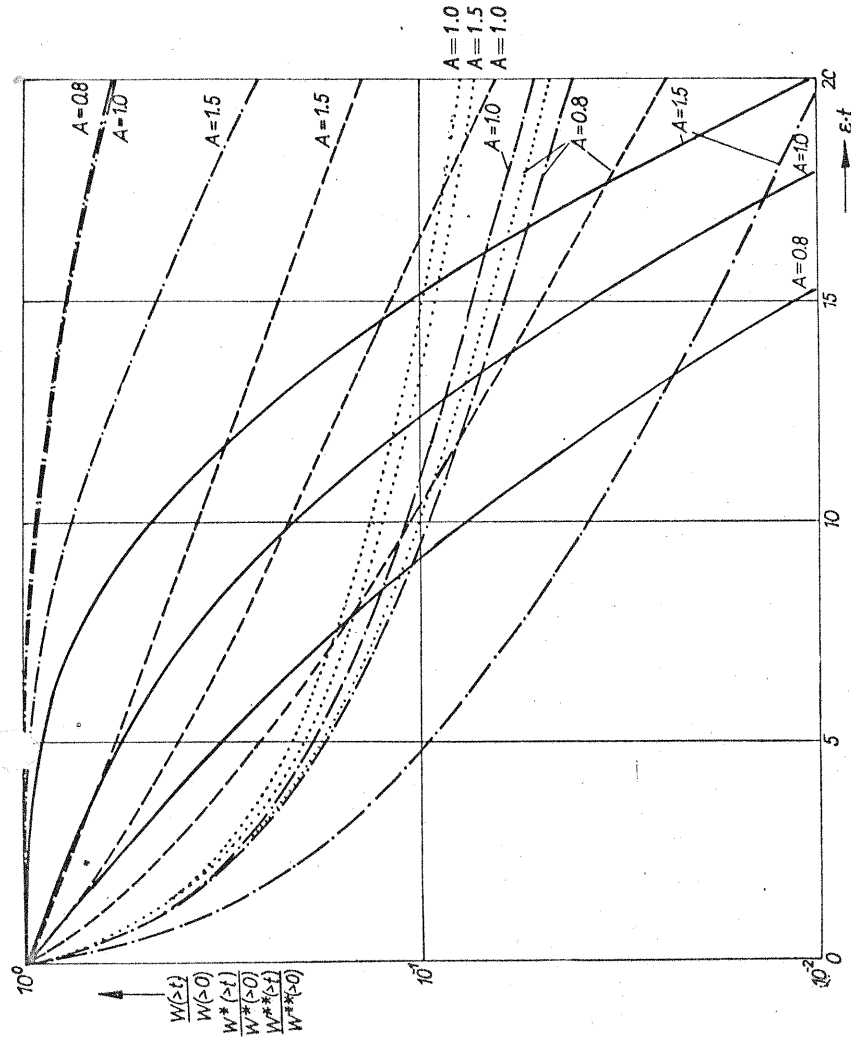$$(82c) \quad \mu M_K(s_1 - z - 1, z) - \mu M_K(s_1 - z - 2, z) = K M_{K-1}(s_1 - z - 1, z).$$

The total first and second moments are derived from eq. (81) accordingly:

$$(85)\quad M_1 = p(n; 0)\, \frac{1}{\mu}\, \frac{1-\varrho^{s_1}}{1-\varrho} \left[\frac{1}{1-\varrho} - \frac{s_1 \cdot \varrho^{s_1}}{1-\varrho^{s_1}} \right],$$

$$(86)\quad M_2 = p(n; 0)\, \frac{2}{\mu^2}\, \frac{1}{(1-\varrho)^2} \left[\frac{1-\varrho^{s_1} + \varrho^{2s_1+2} - \varrho^{s_1+2}}{(1-\varrho)^2} - s_1(s_1+2) \cdot \varrho^{s_1} \right].$$

The mean waiting time referred to all waiting calls, $t_W = M_1/W$, agrees with (19a).
The second moment M_2 exceeds the corresponding ones for D1 and D2.

7. NUMERICAL RESULTS

For the queueing system with $n = 1$ and $s_1 = 12$ the distribution functions of waiting time are shown in Fig. 4 for the values $A = 0.8$ (normal load), $A = 1.0$, and $A = 1.5$ (overload) and for the four queue disciplines D1, D2, D3a, and D3b.

Fig. 5 shows the location of the eigenvalues on the negative-real axis together with the limit intervals. The eigenvalues have been calculated, as far as they are not known explicitly, according to an iterative procedure given by Eberlein and
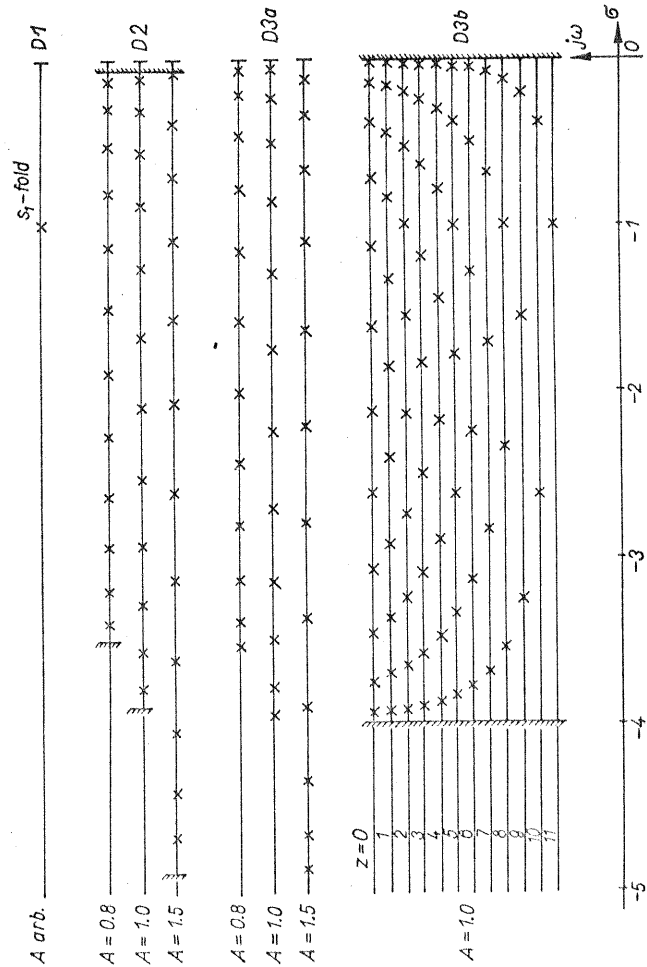

TABLE 1

Moments of the distribution density function of waiting time ($n = 1$, $s_1 = 12$)

Moments referred to waiting calls (D3a: successful waiting calls)	Traffic offered A	Queue disciplines			
		D1	D2	D3a	D3b
M_1/W (M_1^*/W^*)	0.8	$4.114_{10} + 0$	$4.114_{10} + 0$	$3.708_{10} + 0$	$4.114_{10} + 0$
	1.0	$6.500_{10} + 0$	$6.500_{10} + 0$	$4.167_{10} + 0$	$6.500_{10} + 0$
	1.5	$1.009_{10} + 1$	$1.009_{10} + 1$	$1.896_{10} + 0$	$1.009_{10} + 1$
M_2/W (M_2^*/W^*)	0.8	$2.963_{10} + 1$	$4.670_{10} + 1$	$7.994_{10} + 1$	$1.150_{10} + 2$
	1.0	$6.067_{10} + 1$	$1.019_{10} + 2$	$1.050_{10} + 2$	$3.943_{10} + 2$
	1.5	$1.168_{10} + 2$	$2.084_{10} + 2$	$1.856_{10} + 1$	$3.978_{10} + 3$
M_3/W (M_3^*/W^*)	0.8	$2.833_{10} + 2$	$9.182_{10} + 2$	$3.483_{10} + 3$	$7.014_{10} + 3$
	1.0	$6.825_{10} + 2$	$2.546_{10} + 3$	$5.107_{10} + 3$	$5.323_{10} + 4$
	1.5	$1.450_{10} + 3$	$6.488_{10} + 3$	$3.915_{10} + 2$	$5.530_{10} + 6$

Boothroyd [7], where unsymmetrical matrices are transformed to diagonal form by elementary two-dimensional rotations and shears.

In Table 1 the first three moments are listed for demonstration of the effect of the queue disciplines on the shape of the distribution functions of waiting time.

REMARK. Different methods for approximate solution of large systems of differential equations on the basis of exponential sums matching the first four moments as well as by successive series expansions have been reported by the author [4].

The investigations of combined delay and loss systems have also been extended to systems with several input queues and fully as well as limited accessible servers [8, 9].

REFERENCES

[1] R. Syski: Markovian queues. In: Proc. of the Symposium on Congestion Theory. Edited by W. L. Smith and W. E. Wilkinson. The University of North Carolina Press, Chapel Hill 1964, 170—227.

[2] L. Takács: Introduction to the theory of queues. Oxford University Press, New York 1962.

[3] H. Störmer: Wartezeitlenkung in handbedienten Vermittlungsanlagen. Arch. d. Elektr. Übertr. (A.E.Ü.) 10 (1956), 2, 58—64.

[4] P. Kühn: Über die Berechnung der Wartezeiten in Vermittlungs- und Rechnersystemen. Diss. University of Stuttgart, Stuttgart 1971.

Fig. 5. Distribution of eigenvalues on the negative-real axis ($n = 1$, $s_1 = 12$).

[5] R. ZURMÜHL: Matrizen und ihre technischen Anwendungen. 〈 〉th edition. Springer-Verlag, Berlin—Göttingen—Heidelberg 1964.

[6] M. SEGAL: A multiserver system with a finite number of sources and delayed requests served at random. In: Proc. of the 6th International Teletraffic Congress (ITC), Munich 1970, 332/1—4.

[7] P. J. EBERLEIN, J. BOOTHROYD: Solution to the eigenproblem by a norm reducing Jacobi type method. Numer. Math. *11* (1968), 1—12.

[8] P. KÜHN: Parallel waiting queues in real-time computer systems. a) Proc. of the International Computing Symposium, Bonn 1970. ACM-Conference, Part I, 143—158. b) Nachrichtentech. Zeitschr. (NTZ) 23 (1970), 11, 576—582.

[9] P. KÜHN: Combined delay and loss systems with several input queues, full and limited accessibility. a) Proc. of the 6th International Teletraffic Congress (ITC), Munich 1970, 323/1—7. b) Arch. f. Elektronik und Übertragungstech. (A.E.Ü.) 25 (1971), 9/10.

UNIVERSITY OF STUTTGART

INSTITUTE OF SWITCHING AND DATA TECHNICS