OPTIMUM LINK SYSTEMS

by Alfred Lotze

Technical University Stuttgart, Stuttgart, Germany

1. Abstract of the CPE-Method

For the synthesis of "optimum link systems", having a minimum number CPE of crosspoints per erlang, simple formulae are derived. They yield the structural parameters, such as number of stages S; accessibilities $k_1, \dots k_j, \dots k_s$; carried load per linkline; effective accessibility of the link arrangement as a whole.

This derivation doesn't require any approximate loss formula for the minimization as it is known from former publications |1|, |2|. Merely two characteristic quantities have to be prescribed, namely the carried traffic a_1 per inlet of the 1st stage and the socalled "transparency T of the system".

Chapter 2 explains and summarizes all necessary definitions. Chapter 3 compiles the formulae applied to the calculation of the optimum structural parameters. The method holds true for link arrangements with or without gradings between the stages. If the total number of outlets in the last stage remains constant, the number of outgoing trunk groups to which these outlets are divided up is without influence to the optimization. Optimum structural parameters can be calculated not only for the optimum number of stages S_{opt} of the link arrangement, but also for any other chosen number S.

Examples of calculation for link systems without concentration are handled in chapter 4 and for systems with concentration in chapter 5.

Chapter 6 shows the outline of the differentiation method by which the optimum structure formulae were derived. Finally a brief survey about the calculation of call congestion is given in chapter 7.

2. Definitions

2.1 The Number of Crosspoints per Erlang

In a link arrangement having S stages it holds

$$CPE = \frac{k_1}{a_1} + \frac{k_2}{a_2} + \dots + \frac{k_j}{a_j} + \dots + \frac{k_s}{a_s} \qquad (1)$$

being

k_j = accessibility of stage No j

a_1 = carried traffic per inlet of the 1st stage

a_2 = carried traffic per inlet of the 2nd stage

and so forth up to

a_s = carried traffic per inlet of the last stage No S

y_j = carried traffic at k_j outlets.

2.2 The Transparency T

As a new useful measure characterizing a link arrangement is defined:

$$T = (k_1 - y_1) \cdot (k_2 - y_2) \cdot \dots \cdot (k_j - y_j) \cdot \dots \cdot (k_{s-1} - y_{s-1}) \cdot k_s \qquad (2)$$

or

$$T = k_1 \cdot (1 - a_2) \cdot k_2 \cdot (1 - a_3) \cdot \dots \cdot k_{s-1} \cdot (1 - a_s) \cdot k_s \qquad (3)$$

Eq. (2) and (3) demand, that at most one link line exists between two multiples of successive stages.
This condition is obviously correct, because it yields the maximum value of T for constant parameters $k_1, \dots k_j, \dots, k_s$ and $a_1, a_2, \dots, a_s$.
The transparency T says how many free paths "one can see" on the average up to stage No S, multiplied with k_s, from each free inlet of the first stage. Therefore in "fan-out-arrangements" (see section 2.5) T corresponds to the average total accessibility k_m^* with respect to all outlets of the last stage, whatever outgoing trunk group they belong to.

In case of fan-out-structure holds always

$$T = k_m^* = \sum \text{all individual } k_m \text{ of all outgoing trunk groups} \qquad (4)$$

Being n_{out} the total number of outlets in the last stage, in case of fan-out structure holds

$$T < n_{out} \qquad (5)$$

for each carried traffic greater than zero.

2.3 Capacity Index Q

The structural formulae in chapter 3 become easy to handle if we define

$$Q = \frac{T}{a_1} \qquad (6)$$

with T according to eq.(2),(3) and with a_1 being the incoming carried traffic per inlet of the 1st stage.

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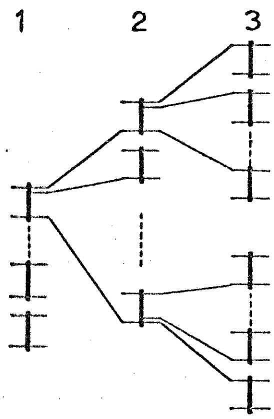
2.4 Total Incoming Traffic Y_{tot}

A link arrangement having n_{in} inlets in the first stage and the carried traffic a_1 per inlet, has

$$Y_{tot} = n_{in} \cdot a_1 \qquad (7)$$

2.5 Link Arrangements with Fan-Out-Structure

Fan-out-structure exists if <u>at most</u> one path can connect a certain inlet of the 1st stage to one arbitrary outlet of the last stage.



Link Arrangement with Fan-Out Structure having 3 Stages

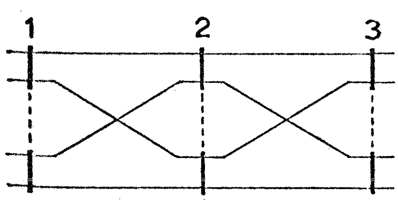
Fig.1

2.6 Link Arrangements with Meshed Structure

Meshed structure exists, if more than exactly one path leads from each individual inlet of the 1st stage to each out of all m_s multiples of the last stage. Therefore

$$T \gtreqless m_s \cdot k_s = n_{out} \qquad (8)$$

is possible, depending on the carried traffic.



Link Arrangement with Meshed Structure having 3 Stages

Fig.2

We define a <u>Meshing Coefficient M</u>

$$M = \frac{T}{n_{out}} \gtreqless 1 \qquad (9)$$

M yields - for a given carried traffic - the average number of <u>free</u> paths between each individual inlet of the 1st stage and each out of all multiples in the last stage.

Having M>1 the probability of socalled "point to point blocking" between one multiple of the 1st stage and any multiple of the last stage practically becomes zero.

2.7 Types of Offered Traffic

<u>Poisson-input</u>, having constant call intensity independent of the number of busy inlets, shall be named |4| "<u>P</u>ure <u>C</u>hance <u>T</u>raffic type No <u>1</u>", abbreviated <u>PCT1</u>.

<u>Engset-input</u>, having a constant call intensity α per idle source, but an overall call intensity depending on the number of idle sources shall be named "<u>P</u>ure <u>C</u>hance <u>T</u>raffic type No <u>2</u>", abbreviated <u>PCT2</u>.

3. Formulae for the Synthesis of Link Arrangements having CPE_{min}

The following structural and traffic parameter formulae are derived in chapter 6. They permit an easy synthesis of link arrangements having the minimum of crosspoints per erlang.

In practice we will obtain slight deviations from this theoretical optimum, because the numbers k_j must be integer. Furthermore deviations can arise for reasons of divisibility with respect to the number of trunks and on the other hand to the accessabilities k_j. The CPE of any available link arrangement simply can be compared with the theoretical minimum.

3.1 Basic Formulae

Be given the number n_{in} of inlets and n_{out} of outlets, furthermore the carried traffic a_1 per inlet of the first stage and the prescribed transparency T. These data will be sufficient for the evaluation of the following formulae:

The number i_1 of inlets per multiple of the 1st stage and the accessibilities of the 2nd and further stages up to the last stage No S become

$$i_1 = k_2 = \dots = k_j = \dots = k_s = 2 \cdot \sqrt[S]{Q/4} \qquad (10)$$

The optimum load per link line between the stages equals to

$$a_2 = a_3 = \dots = a_j = \dots = a_s = 0.5 \text{ erlang} \qquad (11)$$

The multiples of the first stage get outlets with an accessibility

$$k_1 = \frac{a_1}{a_2} \cdot i_1 \qquad (12)$$

From eq. (10), (11), (12) it follows the minimum number of crosspoints per erlang:

$$CPE_{min} = 2 \cdot S \cdot k_j = 4 \cdot S \cdot \sqrt[S]{Q/4} \qquad (13)$$

The very minimum CPE_{opt} is obtained if the following number of stages can be realized:

$$S_{opt} = \ln\frac{Q}{4} \approx 2.3 \cdot {}^{10}\log\frac{Q}{4} \qquad (14)$$

With S_{opt} you get the smallest possible CPE at all from eq. (14) and (13)

$$CPE_{opt} = 4 \cdot e \cdot \ln Q/4 \approx 25 \cdot {}^{10}\log Q/4 \approx 11 \cdot S_{opt} \quad \dots \; e = 2{,}718 \dots \qquad (15)$$

The lower bound $S_{opt} = 1$ is true but for the rather unrealistic case, where $Q = 4e \approx 11$.

The influence of S on CPE_{min} can be seen in fig. 3:

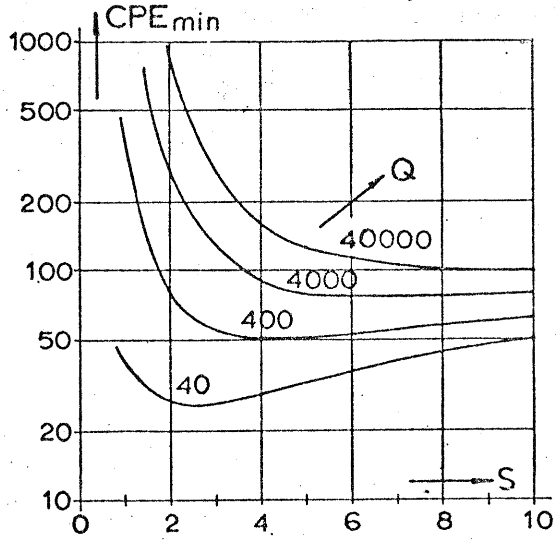


Fig.3

The functions $CPE_{min} = f(S)$ have rather flat minimums. The overall costs per erlang including also the costs of common control in many cases will have their minimum point for $S < S_{opt}$.

3.2 Inlets i_j and Outlets k_j per Multiple in Stage No j

In all stages, where 0.5 erlang per incoming and outgoing links can be realized, obviously holds $i_j = k_j$. The following formulae merely result from the selfevident condition "incoming = outgoing traffic" in these cases where for reasons of practical realization $i_j = k_j$ becomes impossible; or on the other hand because of necessary traffic concentration. The following formulae hold:

Stages without Grading of their Outlets

$$i_1 = k_1 \cdot \frac{a_2}{a_1} \quad ; \quad i_2 = k_2 \cdot \frac{a_3}{a_2} \quad \text{etc.} \qquad \text{cf.} \begin{Bmatrix} 16 \\ 12 \end{Bmatrix}$$

Stages whose Multiple-Outlets are graded

Be a_j the carried traffic per incoming link (per inlet) and be a_{j+1} the carried traffic per outgoing link (behind the grading!). Furthermore be m_j the number of multiples in stage No j each having k_j common outlets, and be H_j the grading ratio. It holds

$$H_j = \frac{m_j \cdot k_j}{\text{number of the grading's outgoing links}} \qquad (17)$$

Number of inlets per multiple

$$i_j = \frac{a_{j+1} \cdot k_j}{a_j \cdot H_j} \geqq 1 \qquad (18)$$

For $i_j = 1$ we obtain $H_{j\,max}$, yielding the best possible balance of outgoing traffic among all outlets of the grading.

Outlets of the grading = inlets of the next stage

$$= \frac{i_j \cdot a_j \cdot m_j}{a_{j+1}} \qquad (19)$$

Gradings do not influence the crosspoints per erlang of the considered stage. However the increase of a_{j+1} by the grading saves crosspoints in the next stage.

3.3 Structure of Type "Fan-Out" or "Meshed"

If $\quad M = \frac{T}{n_{out}} \geqq 1$

we obtain by means of the calculated structural parameters according to eq. (10) to (15) a meshed arrangement.

If $M < 1$, the decision about "meshed" or "fan-out" structure depends on the calculated structural parameters (see examples 4.1). Obviously each link arrangement has fan-out-structure for $S = 2$.

4. Examples for the Calculation of Optimum Parameters in Case of Link Arrangements without remarkable Concentration (but in the last stage)

4.1 Example No 1

The prescribed parameters be

$n_{in} = n_{out} = 200$; $a_1 = 0.6$ erlang; $T = k_m^* = 100$.

Therewith $\quad Q = T/a_1 \approx 167$

and $\quad Q/4 \approx 41.7$

Firstly we get by means of the formulae in section 3 the theoretical values

$S_{opt} = 2.3 \cdot {}^{10}\log \frac{167}{4} = 3.73$ and $CPE_{opt} \approx 40.5$.

Be chosen $S = 4$.

Then we obtain

$i_1 = k_2 = k_3 = k_4 = 2 \cdot \sqrt[4]{41.75} \approx 2 \cdot 2.54 = 5.08$

Furthermore holds

$k_1 \doteq \frac{a_1}{a_{2opt}} \cdot i_1 = \frac{0.6}{0.5} \cdot 5.08 = 1.2 \cdot 5.08 = 6.1$

Now we have to look for a realizable arrangement using integer numbers for i and k. (In practice also available types of selectors will be taken into account.)

4.1.1 Solution A

	Stage No 1		No 2		No 3		No 4	
$n_{in} = 200$ →	5|6	240	5|5	240	5|5	240	6|5	$n_{out} = 200$ →
	40		48		48		40	

With $a_1 = 0.6$ and $a_2 = 0.5$ erlang we get the transparency

$$T = 6\cdot(1-0.5)\cdot 5\cdot(1-0.5)\cdot 5\cdot(1-0.5)\cdot 5$$
$$= 750\cdot 2^{-3} = 93.65 \quad \text{being slightly} < 100$$

and

$$CPE = \frac{k_1}{a_1} + \frac{k_2+k_3+k_4}{a_2} = \frac{6}{0.6} + \frac{15}{0.5} = 40 < 40.5 = CPE_{opt}$$

because of T < 100.

4.1.2 Solution B

being changed

Stage		No 1	No 2	No 3	No 4		
n_{in} = 200		5|6	5|6	6|5	6|5	n_{out} = 200	
——→		240	288	240		——→	
		40	48	48	40		

The transparency gets

$$T = 6\cdot(1-0.5)\cdot 6\cdot(1-0.417)\cdot 5\cdot(1-0.5)\cdot 5$$
$$= 131 > 100$$

and

$$CPE = \frac{6}{0.6} + \frac{6}{0.5} + \frac{5}{0.417} + \frac{5}{0.5} = 42$$

Solution B corrects T from 94 to 131 by means of the small increase from 40 to 42 CPE.

4.1.3 Solving example No 1 with S = 2

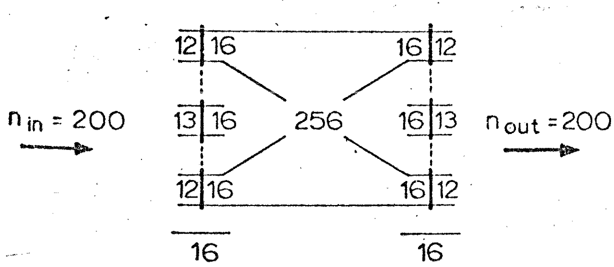
Now S = 2 may be chosen to get a very simple common control. In this case we get the following structural parameters:

$$i_1 = k_2 = 2\sqrt[2]{41.7} \approx 12.92$$

$$k_1 = \frac{a_1}{a_2}\cdot i_1 = \frac{0.6}{0.5}\cdot 12.92 = 15.52$$

Solution C

Be chosen $i_1 = k_2$ = 12 and 13 alternately, and k_1 = 16



We get

$$a_{2mean} = \frac{200\cdot 0.6}{16\cdot 16} = 0.4685 \text{ erlang}$$

and

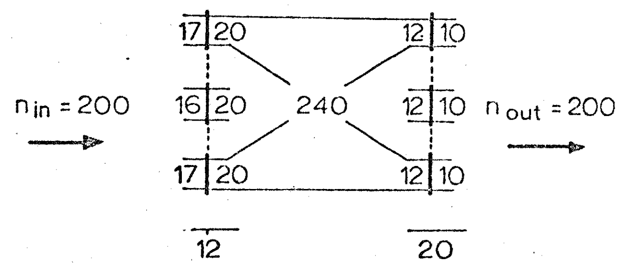
$$T = k_1\cdot(1-a_2)\cdot k_2$$
$$= 16\cdot(1-0.4685)\cdot\frac{12+13}{2} \approx 106$$

$$CPE = \frac{16}{0.6} + \frac{12.5}{0.4685} \approx 53.3$$

Cf. the very minimum (for S = 3.7 and T = 100) is CPE_{opt} = 40.5.

Solution D

Available crossbar selectors, having k=10 and k=20, may be applied.



It becomes

$$CPE = \frac{20}{0.6} + \frac{10}{0.5} \approx 53.3$$

$$T = T_{mean} = 20\cdot(1-0.5)\cdot 10 = 100$$

We can see, that for prescribed number S the differences between the theoretical and the actual values of i_1, k_1 etc. don't influence very much the economy, i.e. the quantity CPE, if the carried traffic is close to the optimum of 0.5 erlang per link line.

4.2. Example No 2

Be given $n_{in} = n_{out}$ = 2000 trunks each, having 0.6 erlang per inlet or outlet respectively.

Full accessibility by means of M > 1 be prescribed, therefore we choose M = 1.5 being far "on the safe side".

Then we get

$$T = M\cdot n_{out} = 1.5\cdot 2000 = 3000$$
$$Q = \frac{T}{a_1} = 5000 \text{ and } Q/4 = 1250$$

The theoretical optimum parameters according to eq. (14) and (15) become

$$S_{opt} = 7.1 \quad \text{and} \quad CPE_{opt} = 77.2.$$

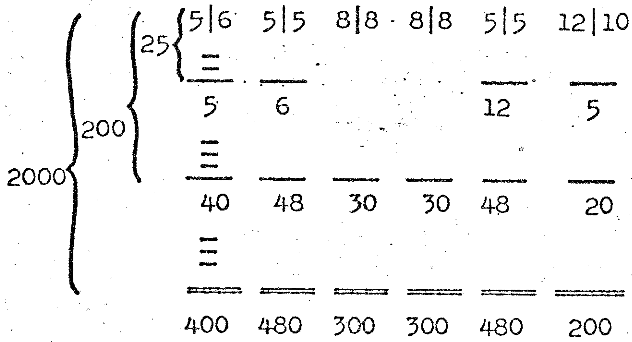
4.2.1 Solution A, using S = 6

S = 6 < 7 may be applied for reasons of a simpler common control.

We find

$$k_2 = k_3 = \ldots = k_6 = 2\cdot\sqrt[6]{1250} \approx 6.6 \text{ and}$$
$$k_1 = \frac{0.6}{0.5}\cdot 6.6 = 7.9$$

The following arrangement is synthesized:



In spite of the fact, that – for reasons of an appropriate link arrangement – parameters k_1, k_j were used, which differ sometimes remarkably from the theoretical ones, the following good result is obtained:

$$T = 6\cdot 5\cdot 8\cdot 8\cdot 5\cdot 10\cdot 0.5^5 = 3000 = M\cdot n_{out}$$

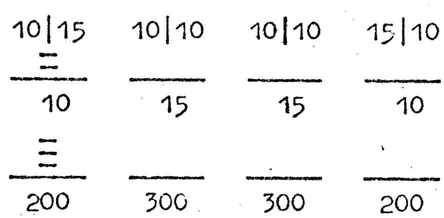
$$CPE = \frac{6}{0.6} + \frac{5+8+8+5+10}{0.5} = 82 \quad \text{(with S=6)}$$

cf. $CPE_{opt} = 77.2$ for $S_{opt} = 7.1$.

This good result has been obtained, because $a_2 = a_3 = a_4 = a_5 = a_6 = 0.5$ erlang is fulfilled. Furthermore the average of $k_2, \ldots, k_6 = 36/5 = 7.2$ is not far from the theory yielding 6.6.

4.2.2 Solution B using S = 4 only

The theory yields $CPE_{min} = 16\cdot\sqrt[4]{1250} \approx 96$ with M = 1.5 and $i_1 = k_2 = k_3 = k_4 \approx 12$; $k_1 \approx 14.4$. We apply merely available selector-types having k = 10.



The actual values become CPE = 100 and T = 3240, i.e. M = 1.62 > 1.5.

4.3 Example for the Planning of a Link System according to Section 4.1.3 with limited Accessibility

Be given n_{in} = 200 lines as well as 4 outgoing groups, having n_i lines each (i = 1, 2,3,4) and the carried traffics Y_i. Furthermore be prescribed the call congestions B_i in case of Poisson input.

From n_i, Y_i and B_i follow the quantities $k_{mi} \approx k_{eff,i}$ by means of the MPJ-table. The transparency results to $T = \sum k_{mi}$. The number of outlets in the last stage is equal to $\sum n_i = 200$. Having calculated by means of T and

$$a_1 = \frac{Y_{tot}}{n_{in}} = 0.6 \text{ erlang}$$

the 2-stage link arrangement according to section 4.1.3*) we get a final stage having 8 multiples with 12 and further 8 multiples with 13 outlets each.

By means of eq. (24)

$$k_{eff,i} \approx k_{mi} = T\cdot\frac{k_{Ri}}{k_2}$$

we can calculate the individual numbers k_{Ri} per each multiple. The results are given in the following table.

i	n_i	Y_i	$B_i \approx$	$k_{mi} \approx$	Outlets k_{Ri} per multiple in the final stage
1	80	52.15	0.0003	42.40	16·5 = 80
2	50	30.45	0.001	26.50	14·3 + 2·4 = 50
3	38	21.0	0.001	20.15	10·2 + 6·3 = 38
4	32	16.4	0.001	16.95	16·2 = 32
-	n_{out} =200	Y_{tot} = 120	-	$\sum$ = T=106	n_{out} = 200

4.4 Planning with Full Accessibility

From the given values $\{n_i, Y_i\}$ follows for full access

$$n_i = k_{mi} \text{ and } T \gtrless \sum k_{mi} = n_{out}$$

$$T \gtrsim 1.3\cdot n_{out}, \text{ i.e. } M = \frac{T}{n_{out}} \gtrsim 1.3 \text{ is recommended.}$$

Obviously the individual numbers k_{Ri} per multiple in the last stage become

$$k_{Ri} = \frac{n_i}{\text{number of multiples in the last stage}}$$

and

$$k_s = \sum_i k_{Ri}$$

5. Examples for the Calculation of Optimum Parameters in Case of Link Arrangements with Concentration in the First Stage (a_1<0.5 erlang)

5.1 Influence of Concentration because of Inlet Blocking

In case of a_1< 0.5 erlang the theory cares for traffic concentration in the 1st stage by means of the formula

$$k_1 = \frac{a_1}{a_2}\cdot i_1 = 2\cdot a_1\cdot i_1 \qquad (12)$$

Eq. (12) effects, that the amount of crosspints per erlang in the 1st stage – in spite of a_1 < 0.5 – equals to that of the following stages. The theory yields with eq. (10), (11), (12)

*) solution C

$$\frac{k_1}{a_1} = \frac{k_2}{a_2} = \;\ldots\ldots = \frac{k_s}{a_s} \qquad (20)$$

In other words: The formula (12) avoids the waste of crosspoints in the 1st (concentration-)stage and pushes the generation of the prescribed transparency T so more to the following stages, so more the inlet load a_1 decreases under 0.5 erlang.

But from $k_1 < i_1$ arises an INLET-BLOCKING probability $[k_1] > 0$, which belongs to the state "all k_1 outlets are busy", where inlets cannot be connected to an idle outlet. The minimization formulae don't take into consideration this "inlet blocking $[k_1]$". Dimensioning k_1 according to eq. (12) in subsciber stages, you get - as a rule - a blocking probability $[k_1]$ which is inadmissible high! In such cases we have to calculate the parameters of the 1st and the following concentration-stages by means of the following policy which may be explained by an example:

5.2 Calculation of the 1st Stage

a) Be prescribed a group of 2000 subscribers, each having - on the average - the busy hour traffic of $a_1 = 0.06$ erlang (sum of outgoing and incoming traffic).

Furthermore a last stage may be necessary, having 160 outlets for two groups of 80 trunks, for incoming and outgoing traffic of 60 erlang each.

Practically full access from any subscriber multiple to the outgoing group as well as full access for the incoming traffic to each subscriber multiple is necessary. Therefore a meshing factor $M \geq 1.5$ may be chosen.

b) Inlets and Outlets of the 1st Stage

Each inlet of the first stage has access to k_1 outlets. If no grading behind the first stage is applied, you have to choose firstly the number i_1 of inlets per multiple such, that - in spite of the well-known statistical variation of the individual subscriber traffics - the actual incoming traffic per multiple doesn't vary too much from the mean $y_1 = i_1 \cdot a_1$. From experience a lower bound is $i_1 \approx 10$.

If a grading behind the first stage (sometimes be named "transposition") is applied, the balance of unequal subscriber traffics becomes improved. Furthermore the admissible load a_2 per link to the 2nd stage is greater than "without grading", if the same upper limit for the inlet blocking $[k_1]$ is prescribed.. Therefore the number of the grading's outgoing links will be smaller than the 1st stage's outlets. The grade of service will be limited by $[k_1]$, mainly with respect to the incoming calls which cannot be connected to an idle subscriber because of $[k_1]$. Therefore only $[k_1] \approx 0.001$ ... will be admissible.

. Because of the large and uneconomic amount of $k_1/a_1 = k_1/0.06$ crosspoints per erlang in the 1st stage we choose k_1 as small as possible with respect to i_1 and $[k_1]$. The optimum load $a_2 = 0.5$ erlang and the small optimum value of k_1 according to eq.(12) cannot be obtained mostly. From $a_1 = 0.06$ erlang, from the chosen number $i_1 = 10$ and from a prescribed upper limit $[k_1] \leq 0.001$ may follow, that $k_1 = 4$ is necessary. (The eq.(12) yields

$$k_1 = \frac{0.06}{0.5}\cdot 10 = 1.2!!)$$

The 1st stage is considered to be "without grading". (This simplifies sometimes the common control.) Up to now the following data are known:

$$i_1 = 10 \quad\Big|\quad a_1 = 0.06 \text{ erlang}$$
$$k_1 = 4 \quad\Big|\quad y_1 = 10 \cdot 0.06 = 0.6 \text{ erlang}$$
$$a_2 = 0.06 \cdot \frac{10}{4} = 0.15 \text{ erlang}$$

The 1st stage's share of the total sum CPE amounts to $4/0.06 \approx 66.7$ crosspoints per erlang.

Using these values we can design the further stages step by step.

5.3 Calculation of the 2nd and further Stages

Step No 1

From $n_{out} = 160$ and $M = 1.5$ follows for the system as a whole

$$T_{min} = 160 \cdot 1.5 = 240$$

The 1st stage's share of T becomes

$$t_1 = k_1 \cdot (1-a_2) = 4 \cdot (1-0.15) = 3.4$$

The further arrangement must effect the rest of T, i.e. $T^* = T/t_1$.

$$T^* = \frac{240}{3.4} \approx 70.6 \text{ and } \frac{Q^*}{4} = \frac{T^*}{a_2 \cdot 4} = \frac{70.6}{0.15 \cdot 4} = 117.8$$

Therewith

$$S^*_{opt} = 2.3 \cdot {}^{10}\log 117.8 = 2.3 \cdot 2.071 \approx 4.75$$

Be chosen $S^* = 3$ only, we get

$$i_2 = k_j = 2 \cdot \sqrt[3]{117.8} = 2 \cdot 4.9 = 9.8$$

$$k_2 = i_2 \cdot \frac{a_2}{a_3} = 9.8 \cdot \frac{0.15}{0.5} = 2.94$$

Be chosen $k_2 = 4$ to get a smaller blocking. According to this value, we can design 100-groups in stage No 1 and No 2:

	No1	No2
100 {	10|4	10|4
	=	
	10	4

The outlets of stage No 2 and so the inlets of stage No 3 carry

$$a_3 = 0.15 \cdot \frac{10}{4} = 0.375 \text{ erlang each.}$$

The average accessibility from one inlet of stage No 1 with respect to the 16 outlets of the 2nd stage becomes

$$k_{1,2} = k_1 \cdot (1-a_2) \cdot k_2 = 3.4 \cdot 4 = 13.6 \;.$$

Before calculating the last 2 stages we have to check, if the "INLET BLOCKING" up to the outlets of the 2nd stage accords to the prescribed upper limit of 0.001.

Considering the 4·4 outlets which carry 100·0.06 = 6 erlang and which are hunted with $k_{1,2} = 13.6$, we find an upper limit for the "inlet blocking" by means of the MPJ-tables [8], [11] calculated for Poisson input. Reading out with

$$\{y = 6 \text{ erlang},\ n = 16 \text{ and } k = k_{1,2} = 13.6\}$$

we obtain

$[k_{1,2}] < 0.0005$, being admissible.

(For yet more exact calculations and with respect to the actually given Engset input we can use the BQ-Formula (see chapter 7).)

Step No 2

The share $t_{1,2}$ of T with respect to the 1st and to the 2nd stage both amounts to

$$t_{1,2} = k_1\cdot k_2\cdot(1-a_2)(1-a_3) = 4\cdot4\cdot0.85\cdot0.625 = 8.5$$

From this follows

$$T^{**} = \frac{T}{t_{1,2}} = 240/8.5 \approx 28.2 \quad \text{and}$$

$$\frac{Q^{**}}{4} = \frac{T^{**}}{a_3\cdot 4} = \frac{T}{t_{1,2}\cdot a_3\cdot 4} = \frac{240}{8.5\cdot 0.375\cdot 4} \approx 19.$$

With $S^{**} = S^{*} - 1 = 2$ we get for a "separate system" consisting of Stage No 3 and 4

$$i_3 = k_4 = 2\sqrt[2]{19} \approx 2\cdot 4.36 = 8.72\ .$$

Be chosen $i_3 = k_4 = 8$. Because of $a_3 = 0.375$ erlang we obtain

$$k_3 = i_3\cdot\frac{a_3}{a_4} = 8\cdot\frac{0.375}{0.5} \approx 6.$$

By trial we can see that because of $k_4 = 8 < 8.72$ the value T becomes too small, therefore $k_3 = 7$ and $a_4 = 0.429$.

The "inlet blocking check" up to the outlets of stage No 3 yields:

$$k_{1,2,3} = k_1\cdot k_2\cdot(1-a_2)(1-a_3)\cdot k_3 = 4\cdot4\cdot0.85\cdot0.625\cdot7 = 59.5$$

and with $\{n = 40\cdot7 = 280;\ Y_{tot} = 120;\ k = 59.5\}$ follows from the MPJ-tables $[k_{1,2,3}] \ll 0.001$.

With $k_3 = 7$ and $a_4 = 0.429$ and $Y_{tot} = 120$ erlang we get $120/(0.429\cdot7) = 40$ multiples in the 3rd stage.

The 320 outlets behind the 2nd stage have to be divided up to the 40 multiples of the 3rd stage. We obtain the following design:

<table>
<tr><td>No 1</td><td>No 2</td><td>No 3</td><td></td></tr>
<tr><td>10|4</td><td>10|4</td><td>8|7</td><td></td></tr>
<tr><td>≡→800</td><td>→320</td><td>≡</td><td>→280</td></tr>
<tr><td>10</td><td>4</td><td></td><td></td></tr>
<tr><td>≡</td><td></td><td>8|7</td><td></td></tr>
<tr><td>200</td><td>80</td><td>40</td><td></td></tr>
</table>

Step No 3

The (last) 4th stage requires 2·80 = 160 outlets, each having 0.75 erlang, i.e. 60 erlang per group. This concentration behind the link arrangement with a factor 0.75/0.5 = 1.5 can be obtained by using 20 multiples each having 14 inlets in the last stage.

The final design becomes:

<table>
<tr><td>No 1</td><td>No 2</td><td>No 3</td><td>No 4</td><td></td></tr>
<tr><td>10|4</td><td>10|4</td><td>8|7</td><td>14|8</td><td>= 2·4</td></tr>
<tr><td>≡→800</td><td>→320</td><td>→280</td><td></td><td></td></tr>
<tr><td>200</td><td>80</td><td>40</td><td>20</td><td></td></tr>
</table>

Now we have to check, if $M \geq 1.5$ has been obtained:

$$T_{final} = k_{1,2,3}\cdot(1-a_4)\cdot 8 = 59.5\cdot(1-0.429)\cdot 8 = 272$$

With $n_{out} = 160$ we get $M = 272/160 = 1.7$

Finally CPE may be calculated

$$CPE = \frac{4}{0.06} + \frac{4}{0.15} + \frac{7}{0.375} + \frac{8}{0.429}$$

$$= 66.7 + (26.7 + 18.7 + 18.7)$$

$$= 66.7 + 64.1 = 130.8$$

Comparison:

Without taking into account the inlet blocking by concentration, S = 4 would yield:

$Q = 240/0.06 = 4000$; $Q/4 = 1000$ and

$$i_1 = k_j = 2\sqrt[4]{1000} \approx 2\cdot5.625 = 11.25;$$

furthermore

$$k_1 = \frac{a_1}{a_2}\cdot i_1 = \frac{0.06}{0.5}\cdot 11.25 = 1.35 \sim 2$$

$$i_1 = \frac{a_2}{a_1}\cdot k_1 = \frac{0.5}{0.06}\cdot 2 \approx 16.67 \sim 16 \text{ or } 17$$

alternately, and therewith $a_2 = a_3 = a_4 = 0.5$ erlang.

<table>
<tr><td></td><td></td><td>No 1</td><td>No 2</td><td>No 3</td><td>No 4</td></tr>
<tr><td rowspan="6">2000</td><td rowspan="3">100</td><td>16|2</td><td>12|12</td><td>12|12</td><td>12|8</td></tr>
<tr><td>≡
17|2</td><td></td><td></td><td></td></tr>
<tr><td>6</td><td></td><td></td><td></td></tr>
<tr><td></td><td>≡→240
16|2</td><td>≡→ 240</td><td>≡→ 240</td><td>≡→ 160</td></tr>
<tr><td rowspan="2">100</td><td>≡
17|2</td><td>12|12</td><td>12|12</td><td>12|8</td></tr>
<tr><td>6</td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td>120</td><td>20</td><td>20</td><td>20</td></tr>
</table>

$$CPE = \frac{2}{0.06} + \frac{12}{0.5} + \frac{12}{0.5} + \frac{8}{0.5}$$

$$= 33.3 + 24 + 24 + 16 = 33.3 + 64$$

$$= 97.5$$

and $T = 1 \cdot 6 \cdot 6 \cdot 8 = 288$

It can be seen, that the appropriate regard to the admissible inlet blockings has increased the crosspoints per erlang but of the 1st stage, whereas the economy of stages No 2, 3, 4 as a whole has not been increased.

As a rule, inlet blocking thoroughly must be taken into account for the dimensioning of the 1st stage. The inlet blocking by concentration in further stages becomes smaller and smaller and can be neglected often.

6. Outline of the Theory

By means of

$$CPE = \frac{k_1}{a_1} + \frac{k_2}{a_2} + \frac{k_3}{a_3} + \dots + \frac{k_s}{a_s} \qquad (1)$$

and

$$T = k_1 \cdot (1-a_2) \cdot k_2 \cdot (1-a_3) \cdot \dots \cdot k_{s-1} \cdot (1-a_s) \cdot k_s \qquad (3)$$

furthermore with

$$Q = \frac{T}{a_1} \qquad (8)$$

follows

$$\frac{k_1}{a_1} = \frac{Q}{k_2 \cdot k_3 \cdot \dots \cdot k_s \cdot (1-a_2) \cdot \dots \cdot (1-a_s)} \qquad (21)$$

Inserting eq. (21) in eq.(1) we get

$$CPE = \frac{Q}{k_2 \cdot \dots \cdot k_s \cdot (1-a_2) \cdot (1-a_3) \cdot \dots \cdot (1-a_s)} + \frac{k_2}{a_2} + \dots + \frac{k_s}{a_s} \qquad (22)$$

By partial derivation we obtain (for prescribed quantities T, S, a)

$$\frac{\partial CPE}{\partial k_2} = \frac{\partial CPE}{\partial k_3} = \dots = \frac{\partial CPE}{\partial k_s} =$$

$$\frac{\partial CPE}{\partial a_2} = \frac{\partial CPE}{\partial a_3} = \dots = \frac{\partial CPE}{\partial a_s} = 0 \qquad (23)$$

Solving this system of $2 \cdot (S-1)$ linear equations we obtain the formulae (10), (11), (12) and (13).

In a second step we consider

$$CPE_{min} = 4 \cdot S \cdot \sqrt[S]{Q/4} \qquad (13)$$

The derivation

$$\frac{dCPE_{min}}{dS} = 0$$

yields

$$S_{opt} = \ln Q/4 \qquad (14)$$

and with eq. (13) the very minimum in case of S_{opt} stages

$$CPE_{opt} = 4 \cdot e \cdot S_{opt} \qquad e = 2.718\dots \qquad (15)$$

7. Remarks to the Calculation of Loss (|5|, |6|, |7|, |8|, |12|, |13|)

7.1 Link Arrangements with M<1 and without Concentration (but in the last stage; cf. chapter 4)

7.1.1 For a certain outgoing trunk group No R, using k_R out of k_s outlets per multiple of the last stage the average limited accessibility holds

$$k_{m,R} = T \cdot \frac{k_R}{k_s} \approx k_{eff,R} \qquad (24)$$

As it is known $k_{m,R}$ is a good approximate value for the effective availability $k_{eff,R}$.

Therefore the call congestion can be calculated like for single stage gradings having a limited accessibility $k = k_{m,R}$ as well as the same carried traffic y_R on the same number n_R of trunks.

7.1.2 In case of PCT1, i.e. of Poisson-input, the MPJ-tables yield call congestions close to reality |7|, |8|, |9|, |10|, |11|.

7.1.3 In case of PCT2, i.e. Engset-input, a new "BQ-Formula", by analogy to the MPJ-Formula, has been developped |12|, |13|. It has been checked and found to be also very close to artificial traffic tests. Details are explained in the Fifth ITC-paper of D. Botsch (Technical University of Stuttgart) "International Standardizing of Loss Formulae".

Applying the BQ-Formula to link arrangements, we must take into account that number of sources, which - on the average - will be at the disposal to offer calls to the considered outgoing trunk group No R:

Be n_{in} the total number of inlets (identical with sources) of the link system and be η_R the fraction of the total traffic $Y_{tot} = n_{in} \cdot a_1$ which is carried from the outgoing group in consideration.

Then the "effective number of sources q_{eff}" is that quantity being on the average not busy by other traffic flows:

$$q_{eff} = n_{in} - n_{in} \cdot (1-\eta_R) \cdot a_1$$

$$= n_{in} \cdot (1-a_1 + a_1 \cdot \eta_R) \qquad (25)$$

7.2 Link Arrangements without Concentration, having Full Accessibility

In case of full accessibility ($M \geq 1$) we apply for PCT1 Erlang's formula $E_{1,n}(A)$.

In case of PCT2 we apply Engset's formula $E_n(q, \alpha)$ using q_{eff} sources of traffic according to eq. (25).

7.3 Link Arrangements with Concentration

7.3.1 Route Blocking [R]

Firstly the part [R] of call congestion may be calculated, which depends on the "Route Blocking", i.e. this part which is caused either by the limited access $k_{m,R}$ to the outgoing group; or in case of full access by the probability that "all n_R trunks of the considered outgoing group are busy". So far the calculation runs exactly as described in 7.1 and 7.2.

7.3.2 Inlet Blocking [K]

In addition to the part [R] another part [K] of loss arises caused by the "Inlet Blocking", which occurs if a hunting inlet of the first stage finds busy
all out of the available k_1 outlets in the 1st stage,
or all outlets of the 2nd stage being available, etc. etc.,
or all outlets up to the last but one stage of the link arrangement.

Stages No 1 up to No S-1 having no concentration of traffic from inlets to outlets don't cause inlet blocking at all (cf. chapter 5, section 5.2 and 5.3).

7.3.3 Calculation of Inlet Blocking [K] for PCT1 (Poisson Input)

a) Stage No 1

If no grading between the multiples exists

$$[k_1] = E_{1,k_1}(A_o), \text{ with } A_o = \frac{k_1 \cdot a_2}{1 - E_{1,k_1}(A_o)} \quad (26)$$

Diagrams and tables see |7|, |8|, |11|.

If a grading exists, having n_1 outlets, each carrying on the average a_2 erlang, the MPJ-formula is applied for a group with $\{n = n_1;\ k = k_1;\ y = n_1 \cdot a_2\}$; |7|, |8|, |9|, |10|, |11|.

A first approximation (lower limit!) for small values k_1 and a_2 will be:

$$[k_1] = a_2^{k_1} \quad (27)$$

b) Stage No 2 and Further Stages

In each case the MPJ-formula can be applied. With n_2 the total number of links from the 2nd to the 3rd stage furthermore with $k_{1,2}$ the average accessible links in the considered case

$$k_{1,2} = k_1 \cdot (1 - a_2) \cdot k_2 ,$$

and with a_3 = carried traffic per link between stage 2 and 3, we use the MPJ-tables for $\{n = n_2;\ k = k_{1,2};\ y = n_2 \cdot a_3\}$ and read out $[k_{1,2}] \equiv$ "B_k" in the tables.

Analogously we use the MPJ-tables for $\{n = n_3;\ k = k_{1,2,3};\ y = n_3 \cdot a_4\}$ to calculate $[k_{1,2,3}]$, where $k_{1,2,3} = k_1 \cdot k_2 \cdot k_3 \cdot (1-a_2) \cdot (1-a_3)$, etc.

7.3.4 Calculation of Inlet Blocking [K] for PCT2 (Engset Input)

a) Stage No 1

A first approximation (with or without grading) yields a lower limit

$$[k_1] = a_2^{k_1} \text{ (exact for } i_1 = k_1) \quad (27)$$

More exact values for $i_1 > k_1$ can be obtained by Engset's formula.

b) Stage No 2 and Further Stages

The average accessibilities $k_{1,2}$, $k_{1,2,3}$... etc. are calculated as in 7.3.3 b.

The following approximation yields a lower limit:

$$[k_{1,2}] = a_3^{k_{1,2}};\ [k_{1,2,3}] = a_4^{k_{1,2,3}};\ \text{etc.} \quad (28)$$

The application of the MPJ-tables as in 7.3.3 b yields an upper limit.
More exact calculation - if necessary - can be obtained by the BQ-Formula.

7.3.5 Total Call Congestion

a) The values $[k_1]$, $[k_{1,2}]$, $[k_{1,2,3}]$ etc. may approximately be added (upper limit):

$$[K] = [k_1] + [k_{1,2}] + \ldots [k_{1,2,\ldots(s-1)}] \quad (29)$$

b) The states of occupation belonging to the blocking values [R] and [K] are assumed to be independent. Therefore the overall call congestion of the considered trunk group No R becomes

$$B_R = [K] + \{1 - [K]\} \cdot [R] \quad (30)$$

This method of separate calculating [K] and [R] is named CIRB:

COMBINED INLET AND ROUTE BLOCKING.

Further details can be found in the "Proceedings No 3 of the Institute of Switching and Data Technics, Technical University Stuttgart" (see |7| and |8|).

Bibliography

|1| H. Akimaru — Optimum Design of Switching Systems
4th ITC London, 1964, Doc.83

|2| Hase — Die Optimierung 4-stufiger RW-Anordnungen mit Hilfe von Datenverarbeitungsanlagen
Monography
Siemens & Halske, München 1965

|3| A. Lotze — Die Optimalsynthese von Linksystemen
Monography
Stuttgart 1965

|4| K. Rohde, H. Störmer — Durchlasswahrscheinlichkeit bei Vermittlungseinrichtungen der Fernmeldetechnik
Mitteilungsblatt für mathem.Statistik, 1953, p. 185-200

|5| D.Kharkevich — An Approximate Method for Calculating the Number of Junctions in a Crossbar System Exchange
Elektrosvyaz' No 2(1959), p.55-63

|6| N. Bininda, A. Wendt — Die effektive Erreichbarkeit für Abnehmerbündel hinter Zwischenleitungsanordnungen
Nachrichtentechnische Zeitschrift 12(1959), p.579-585

|7| A. Lotze — Computation of Time- and Call-Congestion in link systems with two and more selector-stages and with preselection or groupselection according to an approximation method, which is named "Combined Inlet- and Route-Blocking"
Institute for Switching and Data Technics, Technical University Stuttgart (1962) and Proceedings No 3(1963)

|8| A. Lotze, W. Wagner — Table of the Modified Palm-Jacobaeus-Loss-Formula
Institute for Switching and Data Technics, Technical University Stuttgart (1962) and Proceedings No 3(1963)

|9| A. Lotze — Loss Formula, Artificial Traffic Checks and Quality Standards for Characterizing One Stage Gradings
3.ITC Paris (1961), Doc.28

|10| A.Lotze — Verluste und Gütemerkmale einstufiger Mischungen
Nachrichtentechnische Zeitschrift 14(1961), p.449-453

|11| Institute for Switching and Data Technics, Technical University Stuttgart — Tables for Overflow Variance Coefficient and Loss of Gradings and Full Available Groups
Second edition 1966

|12| A. Bächle, U. Herzog — Berechnung unvollkommener Bündel für Zufallsverkehr zweiter Art(1966)
to be published in Nachrichtentechnische Zeitschrift

|13| D. Botsch — International Standardizing of Loss Formulae?
5th ITC, 1967, New York