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Loss Formula, Artificial Traffic Checks and Quality Standards for Characterizing One Stage Gradings

by

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Contents

N o m e n c l a t u r e

A_k	Traffic offered to a group of N lines which is hunted with the limited availability $k < N$
A_o	Traffic offered to a group with full availability
A_t	Traffic offered to o n e selector group of a grading
B_b	Loss ascertained when offered traffic is equally distributed to all selector groups (balanced)
B_k	Calculated probability of loss of a group, hunted with the availability $k < N$
$B_o = E_N(A_o)$	Blocking probability and probability of loss of a full available group of N lines
B_s	Loss ascertained in an artificial traffic test when the distribution of the offered traffic to the selector groups is definitely unequal (sloping)
$\sigma(k, x)$	Probability, that k outlets are blocked by x occupatious in the hunted group
N	Number of lines in the route
p(x)	Probability of the state, "x lines occupied"
S	Unbalance (slope), measure for unequal distribution of offered traffic to the selector groups
$M = \frac{m \cdot k}{N}$	Average interconnection number of a grading

Loss formula, artificial traffic checks and quality standards for characterizing one stage gradings

Survey

1. For the exact calculation of loss in one-stage gradings the arithmetic work goes quickly up into the unlimited - except for very small gradings. Therefore, all telecommunication firms and administrations are using approximation methods. Some of them use O'Dells formula [1].
   The German Federal Post Office e.g. is using diagrams or tables according to [2]. The Erlang Interconnection Formula [3] might also be used. However, it gives mostly loss values which are smaller than the measured losses, because it is derived under idealized conditions. A summary and discussion of the most known methods is contained e.g. in the publications by A. Elldin [4], R. Syski [5] and others.
2. Tables or diagrams for simple reading - or at least interpolating - the loss for **any** triple of values (A, k, N) or (y, k, N) do still not exist - at least not in Germany.

3. The new approximation method.

3.1. The new approximation method described below for the calculation of loss is easy to handle. It leads to a modification of the well known grading loss formula of Palm/Jacobaeus [6,7] and gives calculated values which are in accord rather well - from small up to big losses - with artificial traffic trials on a digital computer.

3.2. The formula can also be used to calculate easily the overflow probability in case of alternate routing for any values of traffic offered or of traffic loaded in the route hunted at first.

Finally, the method seems to be rather useful for the simple determination of uniform quality standards of gradings (v.par. 8).

4. Theory

4.1. The calculation presumes that the probability distribution of the state p (x) of a traffic load y in a route with N lines is not very essentially influenced by the mode of hunting which was used for the generation of this loaded traffic "y". In other words: The distribution function p (x) of a predetermined loaded traffic of a route with N lines is always about the same, no matter whether this loaded traffic was caused by hunting the route with full availability, by hunting with limited availability in one stage, or by hunting in two or more link stages.

4.2. Based on the approximate assumption in paragr. 4.1, one can assume - irrespectively of the availability "k" of the selectors - for a fixed pair of values (y, N) that easily computable distribution p (x) which - by generation of the traffic value "y" by means of full hunting with a (presumed) offered traffic "A_o" - would hold exactly true.

Therefore, for this offered traffic "A_o" we get the equation:

$$A_o = \frac{y}{1 - B_o} \qquad (1)$$

being $B_o = E_N (A_o)$ the loss in case of full available hunting.

The probability distribution for "x" simultaneous occupations in the route with N lines results from the Erlang formula

$$p\ (x) = \frac{\frac{A_o^x}{x!}}{\sum_{\xi = o}^{N} \frac{A_o^{\xi}}{\xi\ !}} \qquad (2)$$

4.3. Let us now presume that the loaded traffic "y" be statistically evenly distributed on all N lines of the route and - furthermore - will be sufficient for the equation (2). About the causes of this even distribution (for instance: an offered traffic very evenly distributed on the selector groups; or unevenly distributed offered traffic and very good grading), no assumptions are being made yet. Just the same, no assumptions are made with regard to the loss which was necessary for the generation of the loaded traffic "y".

4.4. Now let us inquire about the probability that $x \geqq k$ occupations - which might momentarily exist in the route - are arranged on the N lines just in such a way that the "k" outlets of a determined selector group (but any one chosen) are blocked.

Altogether, $\binom{N}{x}$ patterns of the "x" occupations in the trunk are equally possible and also sufficient to par. 4.3. Imagine that the outlets of a fixed selector group are blocked by "k" of the "x" occupations, there still exist

$$\left\{ \begin{matrix} N - k \\ x - k \end{matrix} \right\}$$

equally possible patterns of the "x occupations on N lines". That means that the blocking probability of a selector group is in the state "x" of the route (according to Erlang [3])

$$\sigma(k, x) = \frac{\left\{\begin{matrix} N - k \\ x - k \end{matrix}\right\}}{\left\{\begin{matrix} N \\ x \end{matrix}\right\}} = \frac{\left\{\begin{matrix} x \\ k \end{matrix}\right\}}{\left\{\begin{matrix} N \\ k \end{matrix}\right\}} \tag{3}$$

4.5. Now, the probability of loss for an offered traffic "A_k" - which is hunting the N lines of the route with the availability "k" - will be

$$B_k = \sum_{x=o}^{N} \sigma(k, x) \cdot p(x) \tag{4}$$

and with (2) and (3)

$$B_k = \sum_{x=o}^{N} \frac{\binom{x}{k} \cdot \frac{A_o^x}{x!}}{\binom{N}{k} \cdot \sum_{\xi=o}^{N} \frac{A_o^{\xi}}{\xi!}} \tag{5}$$

$$\boxed{B_k = \frac{E_N(A_o)}{E_{N-k}(A_o)}} \tag{6}$$

being $A_o = \frac{y}{1 - E_N(A_o)}$, where y is given

Dividend and divisor correspond to Erlang's formulas for two full available groups with the same traffic offered "A_o" and the numbers of lines N and (N - k). Erlang's formula is tabulated in [8].

4.6. "A_o, N and k" were given. After having now calculated "y" and the loss of offered traffic "B_k", the result is the <u>actual</u> offered traffic "A_k" which is - in case of an availability $k < N$ - necessary for the generation of the loaded traffic "y" -

$$A_k = \frac{y}{1 - B_k} \tag{7}$$

The bigger the proportion "N/k" is, the bigger will also be "A_k", compared to the offered traffic "A_o" used for the calculation, which would be required for (presumedly) hunting with full availability. For the limiting case of the full available group - i.e. for "k = N" - we get

$$A_k = A_o \quad \text{and} \quad B_k = E_N \ (A_o), \tag{8}$$

which is also proved by equation (6).

5. Comparison with the approximate formula given by C. Palm and C. Jacobaeus

5.1. Jacobaeus has given in [6] an approximate formula for small losses in gradings very similar to the equation(6). It is taken from a work published - in Swedish language - by C. Palm [7]. There, the approximate formula for s m a l l losses reads:

$$B_k = \frac{E_N \ (A_k)}{E_{N-k}(A_k)} \tag{9}$$

The difference between equation (9) and our equation (6) is only the use of the a c t u a l offered traffic A_k in Palm's equation (9) instead of the fictitious offered traffic A_o used in equation (6) (= offered traffic in case k = N). Therefore, (9) gives useful B-values as long as A_k is still $\approx$ y; that means when losses are s m a l l. On the other hand, equation (6) gives useful average values of loss up to extremely high losses.

6. Check of the equation (6) by artificial traffic tests.

6.1. With the aid of an electronic digital computer of the Technical High School in Stuttgart tests with artificial random traffic were made in order to check the accuracy of equation (6). The fundamental principle of this method is described in [9]. For the production of the required pseudo-random-numbers was used the Multiplicative Congruental Method according to Juncosa [10, 11]. Detailed tests about the production of such random numbers, and the questions raised by them, are to be found in an essay published by my collaborator W. Wagner [12].
Figures 1, 4a, 5a, 6a, 7a, 8a exemplify the conformity of equation (6) with test results. Figures 3a ...3e, 4b, 5b, 6b, 7b, 8b show the gradings.

7. Comparison with A. Elldims Studies

In his valuable "Further Studies on Grading with Random Hunting" [4] Elldin presents on Page 235, Table 5.24 comparisons between his exact formula and various approximate values for symmetrical gradings with three selector groups and symmetrical (balanced) loading.
Some of these values are shown in the following table and compared with the approximate values of equation (6). It is seen, that the values of our modified Palm/Jacobaeus formula (6) have only small deviations from the exact formula and seem to give good approximate losses, also from this point of view.

Table I				Formulae and values, quoted from Elldin's Studies [4, table 5.2.4]					
k	N	A Erl	B_k according to equation (6)	Elldin's exact f. E (2.2.1)	Elldin's approx. f. E (3.5.3a)	Erlang's Interconnection-Form. (4.1.11)	Palm-Jacobaeus H (k) (5.2.2)=(9)	O'Dell pure chance traffic: E_p (5.2.5)	Karlsson pure chance traffic: E'_p (5.2.7)
4	6	0,75	0,084 %	0,0949 %	0,0920 %	0,0779 %	0,0845 %	0,1101 %	0,1098 %
		3,0	8,74 %	8,185 %	8,147 %	7,859 %	9,85 %	9,10 %	8,69 %
		6,0	30,6 %	29,96 %	29,94 %	29,69 %	36,8 %	33,0 %	30,5 %
		12,0	57,3 %		57,1 %	57,1 %	65,6 %		
6	9	2	0,091 %		0,1012 %	0,0802 %	0,0907 %		
		4	2,77 %	2,605 %	2,565 %	2,353 %	2,96 %	2,98 %	2,92 %
		6	10,9 %		10,16 %	9,80 %	12,73 %	11,5 %	10,7 %
		10	30,7 %		31,4 %	29,7 %	37,3 %		
10	15	6	0,245 %	0,2622 %	0,2479 %	0,2006 %	0,248 %	0,304 %	0,303 %
		9	3,44 %		3,10 %	2,86 %	3,79 %		
		12	11,4 %	10,64 %	10,58 %	10,23 %	13,7 %	12,08 %	11,16 %
		15	21,0 %		20,1 %	19,75 %	26,0 %		
20	30	15	0,054 %		0,0546 %	0,0421 %	0,0539 %	0,0743 %	0,0733 %
		18	0,515 %		0,461 %	0,397 %	0,531 %	0,572 %	0,563 %
		24	5,63 %		5,01 %	4,31 %	6,60 %	5,87 %	5,48 %
		36	25,5 %		24,8 %	24,7 %	32,3 %		

8. Quality standards for characterizing gradings.

8.1. The loss formula (6) offers itself by its values approaching reality as a simple and exactly defined comparison base for the designation of gradings, all the more as the Palm's Erlang Tables [8] - required for the calculation of B_k - are everywhere available.

The following quality standards are defined:

8.2. The β - value of a grading be

$$\beta = \frac{B_b}{B_k} \tag{10}$$

It shows the factor "β" by which the actual loss B_b - found by means of artificial traffic tests with b a - l a n c e d offered traffic A_k - differs from a loss B_k of a group with the same data (k, N, A_k), calculated according to (6).

If no characteristic curves $\beta = f\ (B_k)$ should be communicated, but only some particular marks, an index is sufficient to indicate the reference loss B_k. For example: β_{10} would mean the β-test - value belonging to B_k = 10%. Fig. 9 shows the β-curves of the gradings of fig. 1 and 3.

8.3. The γ-value of a grading be

$$\gamma = \frac{B_s}{B_k} \tag{11}$$

The value "γ" indicates the factor by which the actual loss - found by means of an artificial traffic test when the offered traffic is unbalanced ("sloping") - differs from the loss B_k, calculated when balanced traffic offered is presumed.

The value "γ" gives good information about the "balancing capacity" of the grading in consideration. The introduction of the characteristic mark "γ" requires an agreement about the "slope" of the offered traffic by which the loss B_s shall be tested.

8.4. Standardized "slope" of offered traffic for γ-tests.

In order to obtain well comparable results for the characteristic γ where different values of offered traffic and different gradings are concerned, the unbalanced ("sloping") distribution of offered traffic must be defined and uniformly prescribed for artificial traffic tests for γ -determination. The easiest way is to prescribe a traffic value - rising in equal steps ΔA from the first to the last selector group - of the offered traffic A_t per each selector group.

As "slope" of the offered traffic, the term is defined

$$S = A_{t_{max}} : A_{t_{min}} \tag{12}$$

For a predetermined value S of the slope and a predetermined number m of the selector groups, the smallest part-offer can be figured out as

$$A_{t_{min}} = \frac{A}{m} \cdot \frac{2}{S + 1} \tag{13}$$

and the "traffic steps" from selector group to selector group as

$$\Delta A = \frac{A}{m} \cdot \frac{(S - 1)}{(S + 1)} \cdot \frac{2}{(m-1)} = A_{t_{min}} \cdot \frac{(S - 1)}{(m - 1)} \tag{14}$$

Now the question comes up which "slope" shall be prescribed for the γ-tests.

With the number m of the selector groups increases also - in practical service - the expected unevenness of the m part-offers $A_{t\,1} \ldots\ldots\ldots A_{t_m}$, that means the "slope" of the offered traffic. The number m increases generally with the number of lines N. Furthermore, it increases about with 1/k because smaller availabilities "k" require bigger interconnection numbers $M = \frac{m \cdot k}{N}$.

Therefore, it seams to be suitable to prescribe for

γ - tests

$$S = C \cdot \frac{N}{k} \qquad (15)$$

and to fix the factor C appropriately. The tables below are showing that C = 1,5 is leading to reasonable values of the slope, and consequently of the offered traffic $A_{t_{min}}$ and $A_{t_{max}}$.

Table II

k = 10 and $M = \frac{m \cdot k}{N} \geqq 2 : 1$, grading with sequential hunting					
N	$S = \frac{1,5 \cdot N}{k}$	$m \geqq \frac{M \cdot N}{k}$	partial offered traffic in % of total traffic A A_t (average) %	$A_{t_{min}}$%	$A_{t_{max}}$%
30	4,5	6	16,66%	6.06	27,26
60	9	12	8,33%	1,66	15
100	15	20	5%	0,625	9,375
150	22,5	30	3,33%	0,283	6,38

Table III

k = 30 and $M = \frac{m \cdot k}{N} \geqq 1,4 : 1$, grading with sequential hunting					
N	$S = \frac{1,5 \cdot N}{k}$	$m \geqq \frac{M \cdot N}{k}$	partial offered traffic in % of total traffic A A_t (average) %	$A_{t_{min}}$%	$A_{t_{max}}$%
60	3	3	33,33	16,66	50,0
100	5	5	20,0	6,66	33,3
150	7,5	7	14,4	3,36	25,2

As the γ-test has to show clearly the balancing capacity of a grading, it will not be commendable - as a rule - to soften the conditions of the γ-test by choosing the factor C smaller than 1,5.

Besides, for β-diagrams and γ-diagrams it has to be indicated whether the artificial traffic test was hunted by sequential or random hunting of the selector's outlets (corresponding to selectors with or without fixed home position).

Fig. 10 illustrates as example the results of the γ-test for the gradings whose loss curves were shown in fig. 1 and 2. A group with N = 40 lines was sequentially hunted with the availability k = 6 and was investigated with 5 various gradings (compare fig. 3a, b, c, d, e).

9. Summary

9.1. A modification of Palm/Jacobaeus loss formula for gradings is discussed. Their evaluation can be done in a simple way by means of two readings in the "Erlang-Tables" [8].
The loss values, calculated with formula (6) are in rather good accord with the average results of artificial traffic tests.

9.2. Quality standards are defined which relate the losses determined in artificial traffic tests to the calculated value of the formula (6). They make possible quality standards and defined comparisons between different gradings.

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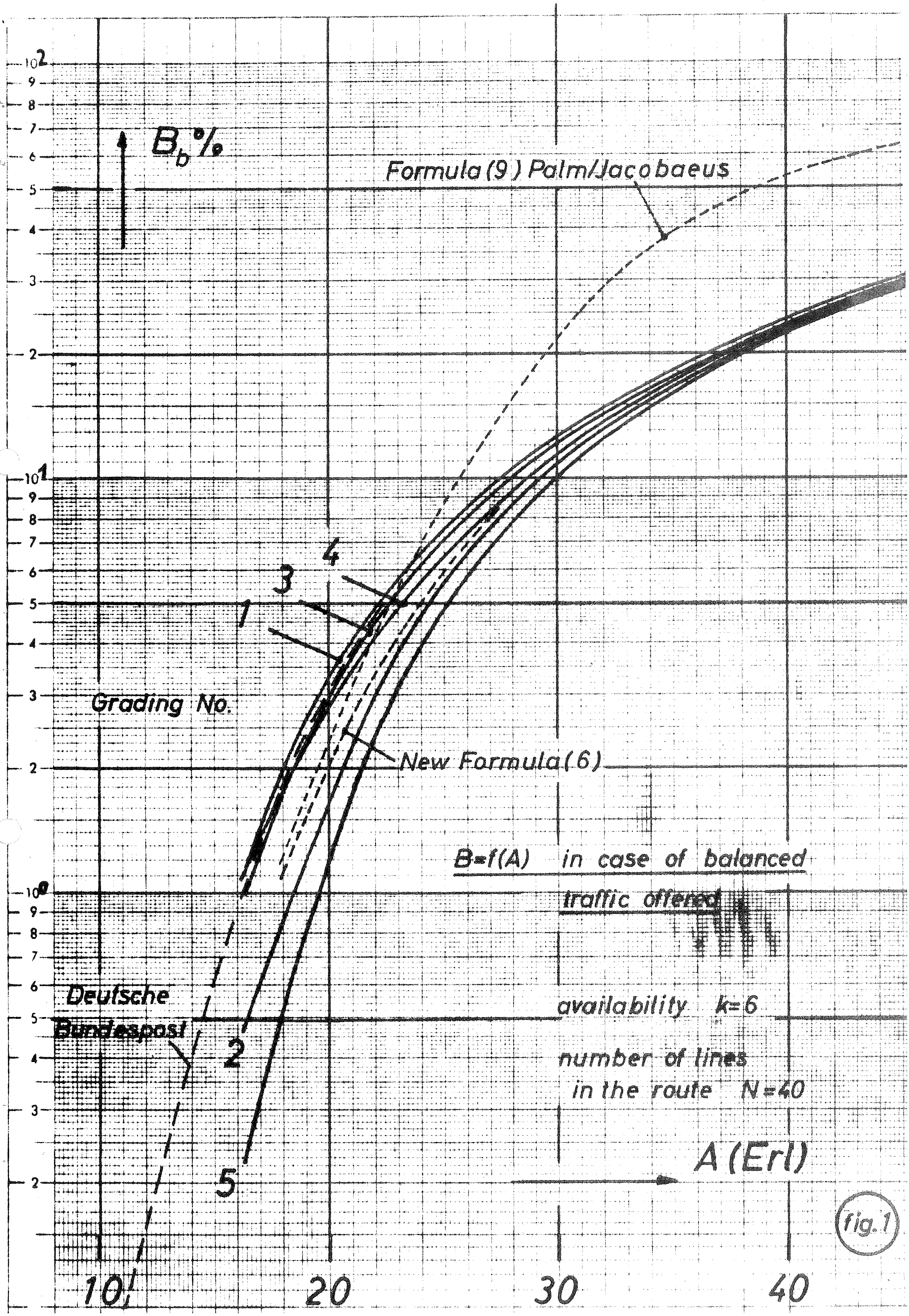
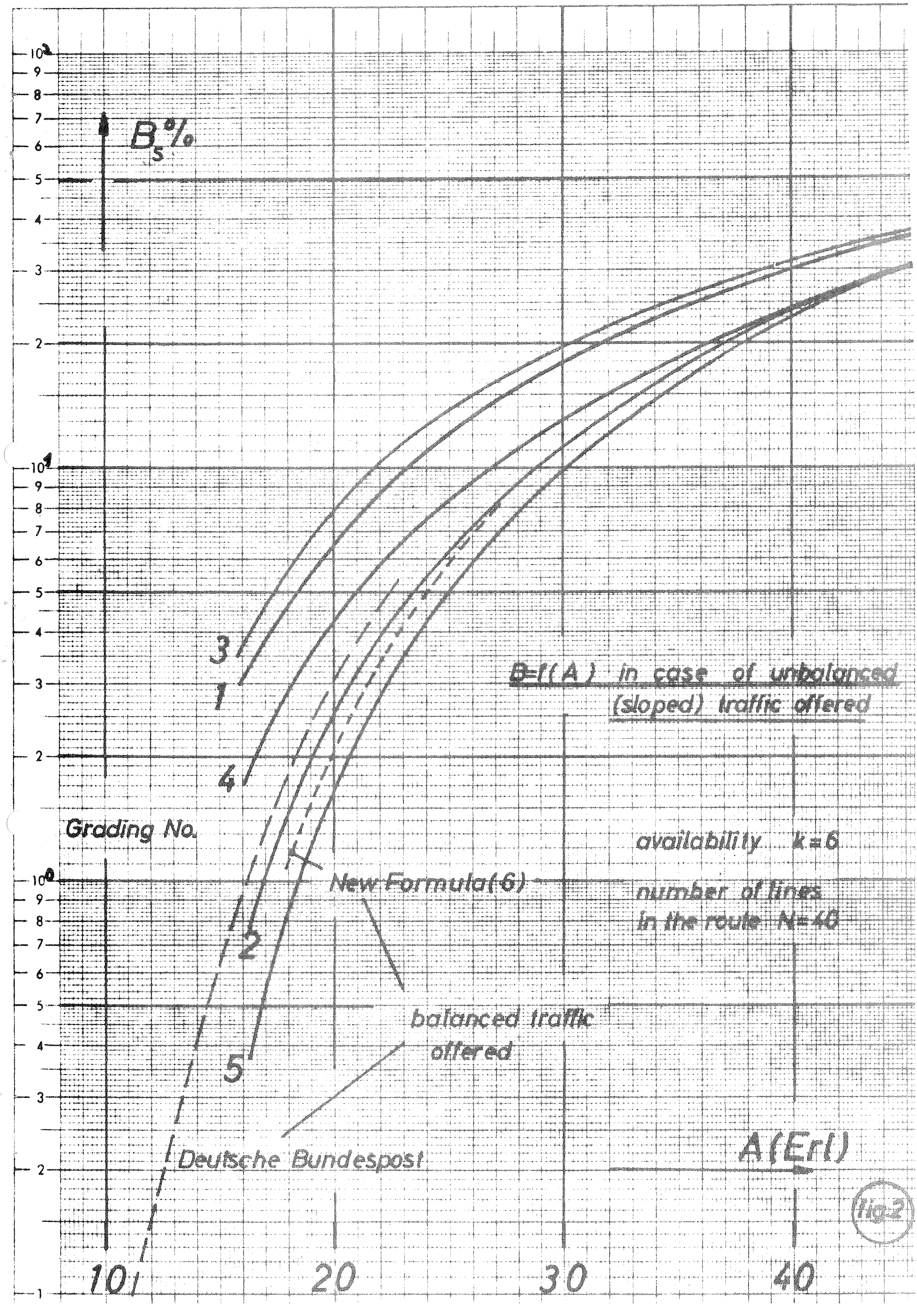
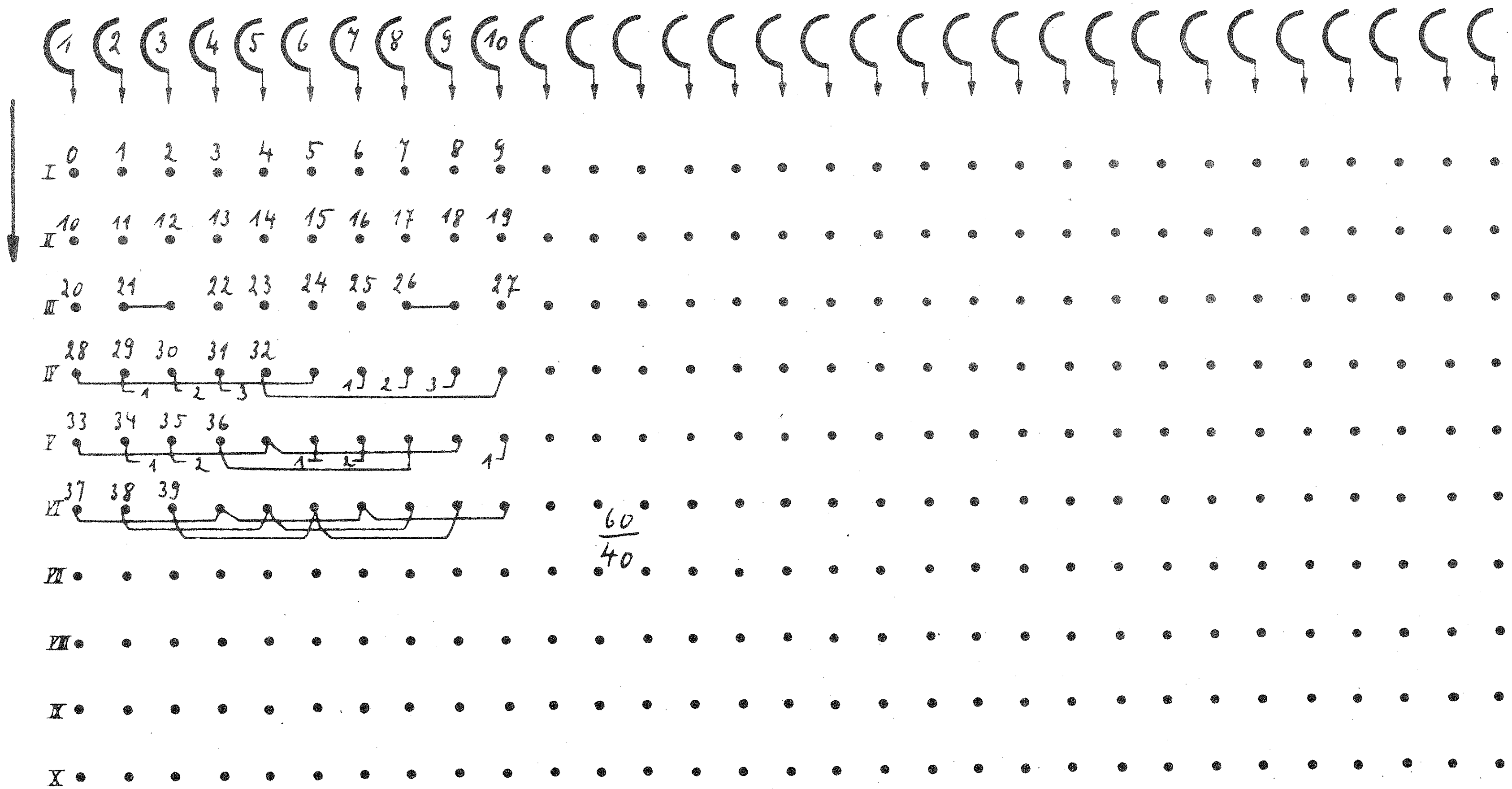


B_b %
Formula (9) Palm/Jacobaeus
Grading No.
1
2
3
4
5
New Formula (6)
B=f(A) in case of balanced traffic offered
Deutsche Bundespost
availability k=6
number of lines in the route N=40
A (Erl)
10
20
30
40
fig. 1



B_s %
3
1
4
2
5
Grading No.
$B=f(A)$ in case of unbalanced (sloped) traffic offered
availability k=6
number of lines in the route N=40
New Formula(6)
balanced traffic offered
Deutsche Bundespost
A(Erl)
10
20
30
40
fig.2



Grading No.1

N=40 k=6

m=10 $M=\frac{60}{40}=1{,}5$

fig3a

21.4.60

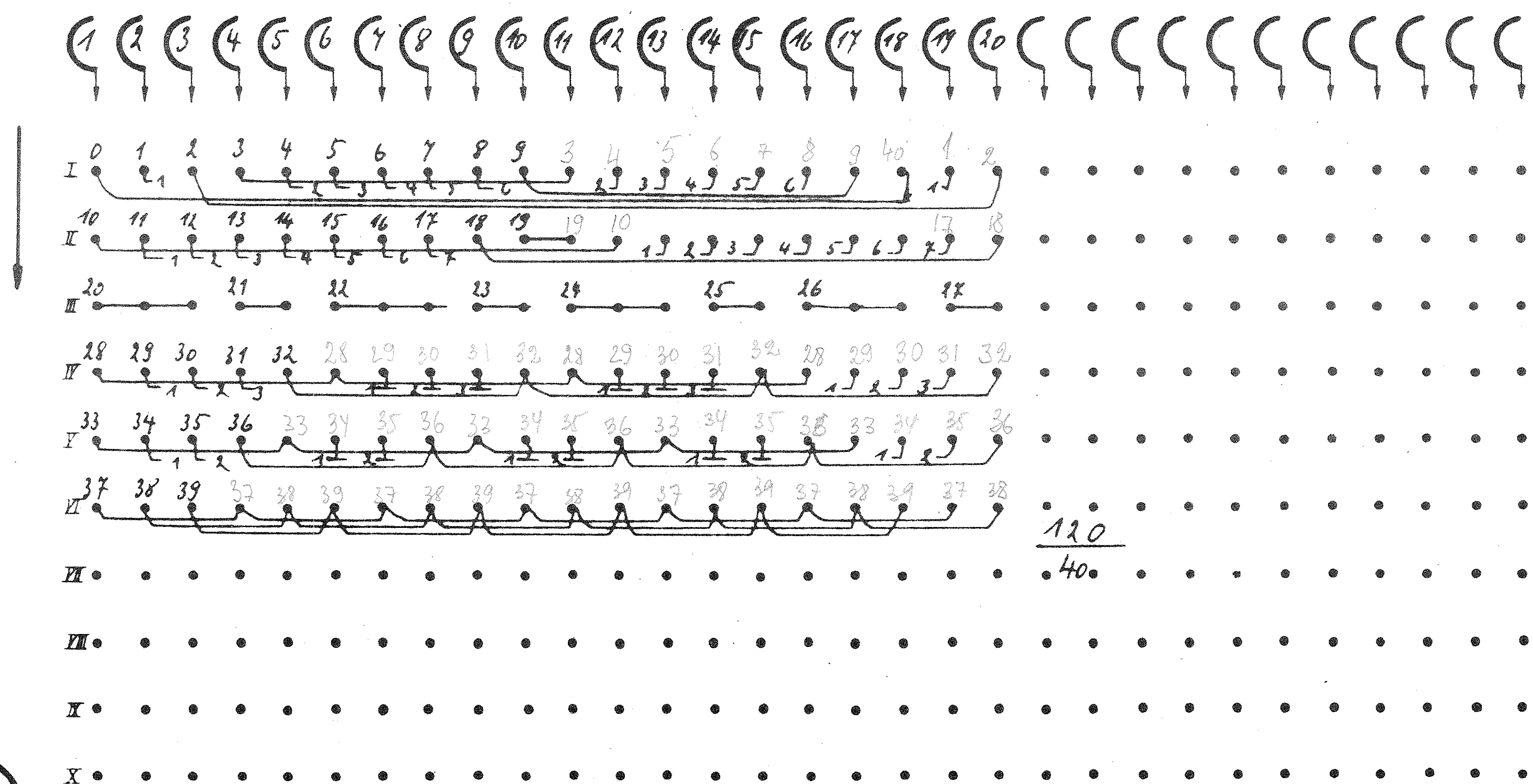


fig.3b

Grading No. 2

$N = 40$ $k = 6$

$m = 20$ $M = \frac{120}{40} = 3$

21.4.60

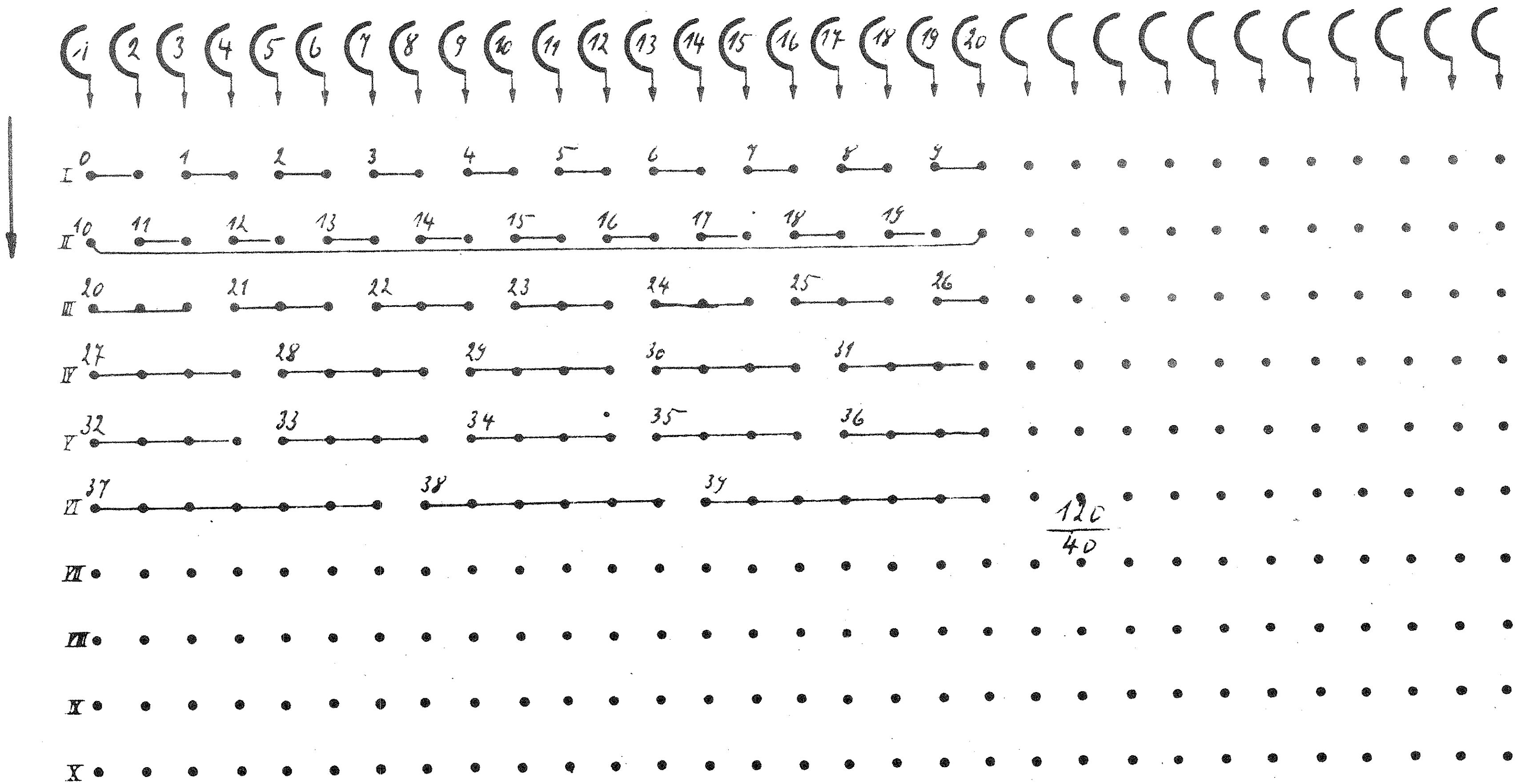


fig.3c

Grading No. 3

$N=40$ $k=6$

$m=20$ $M=\frac{120}{40}=3$

21.4.60

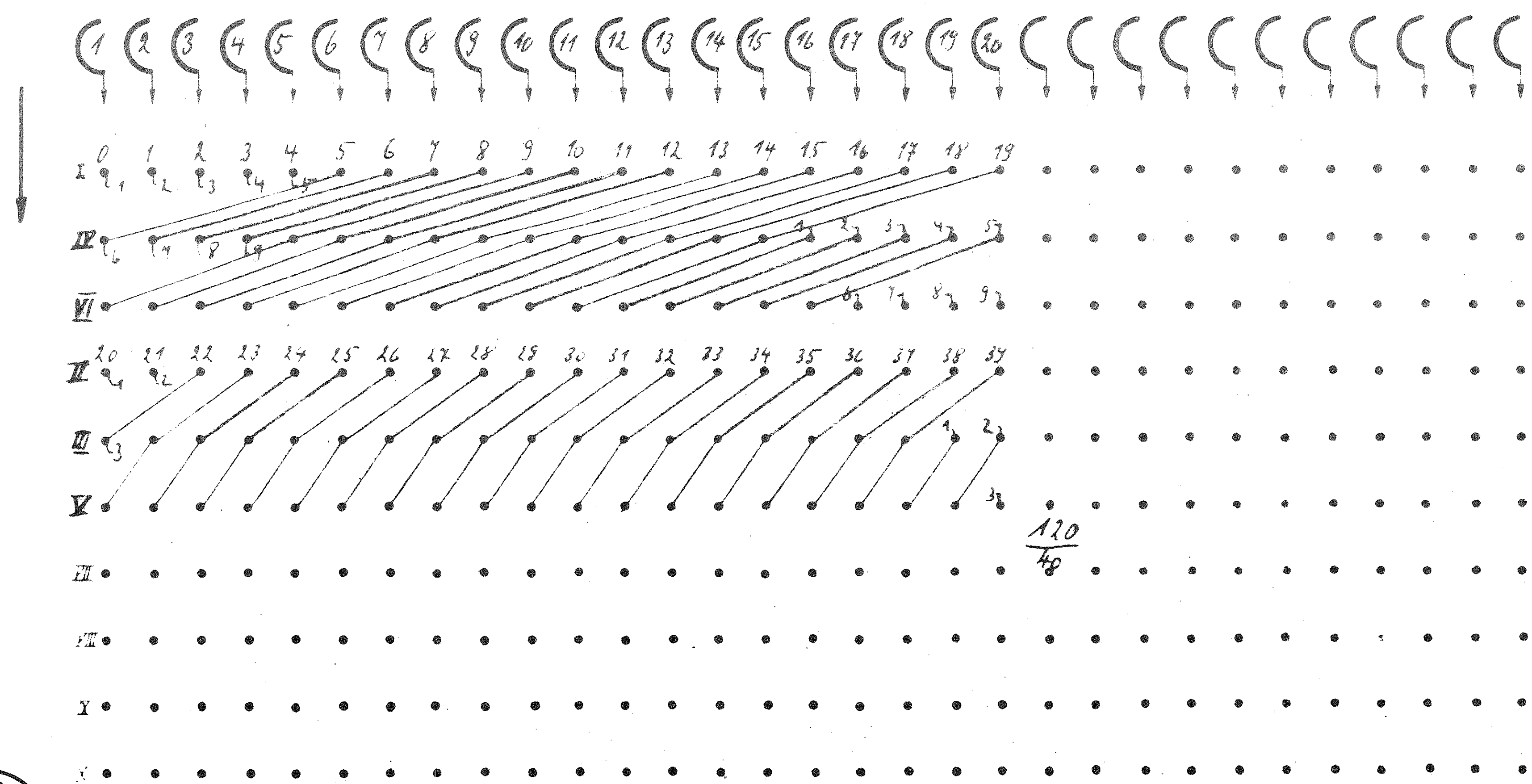


fig. 3d

Grading No.4

$N=40 \quad k=6$

$m=20 \quad M=\frac{120}{40}=3$

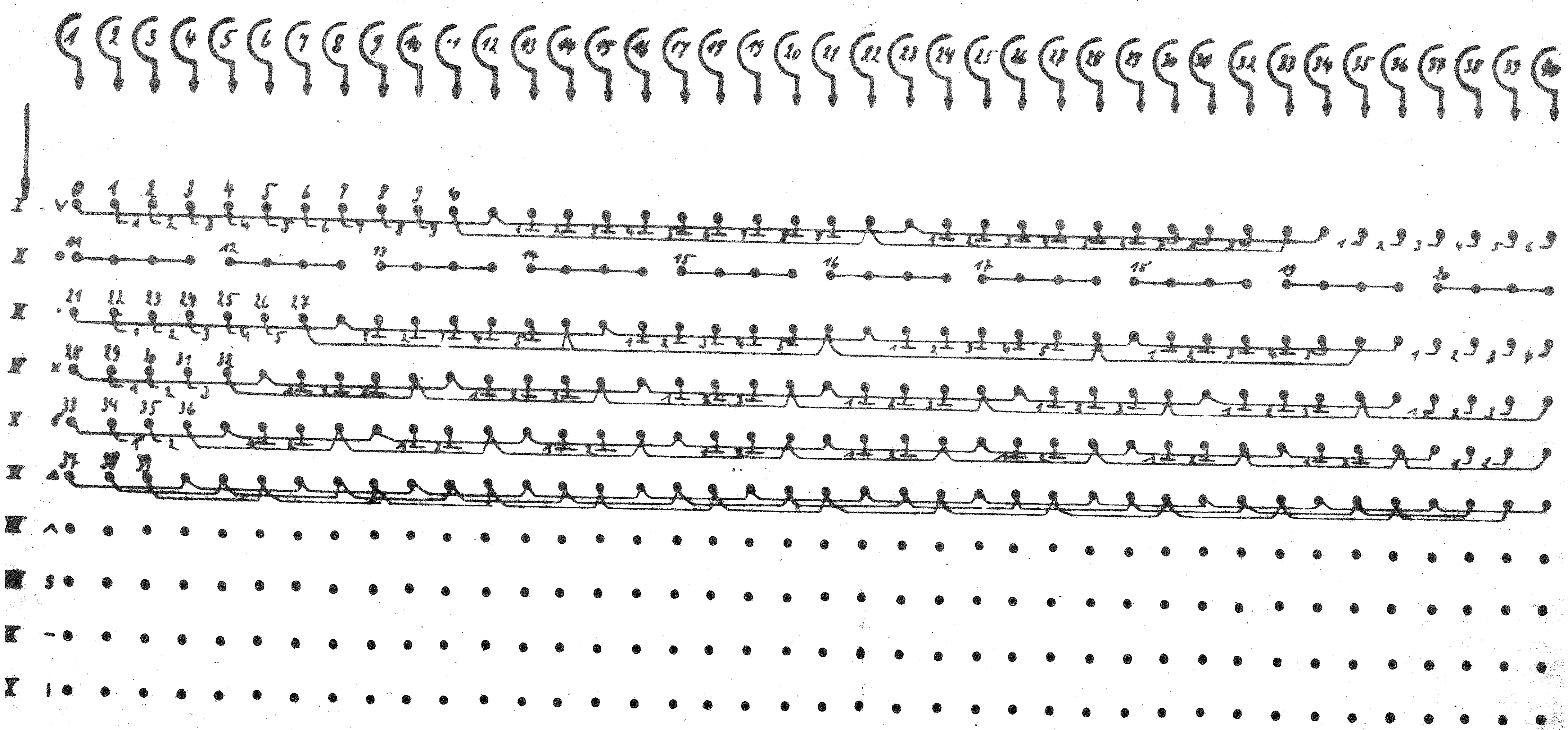
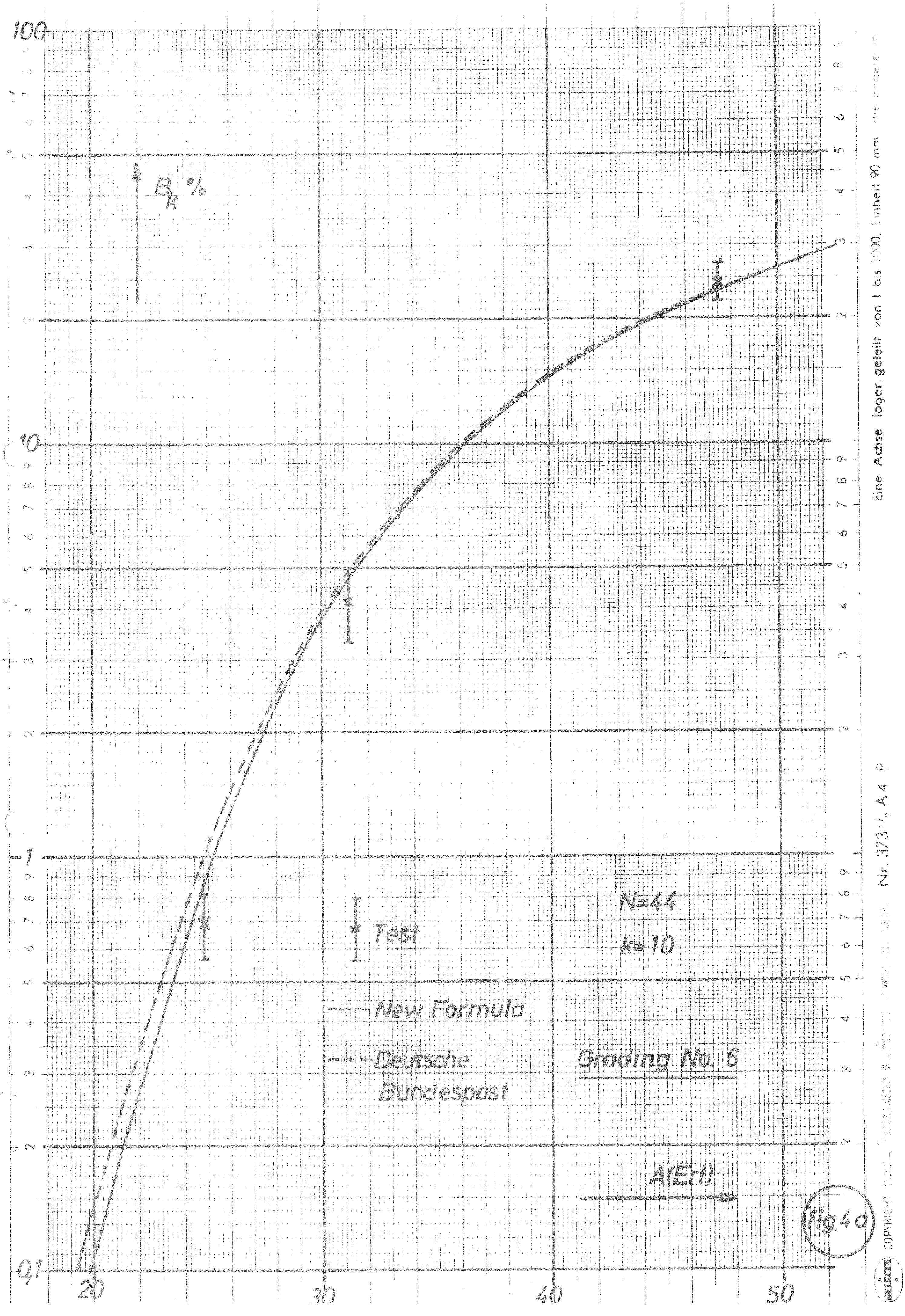


Abb. 36

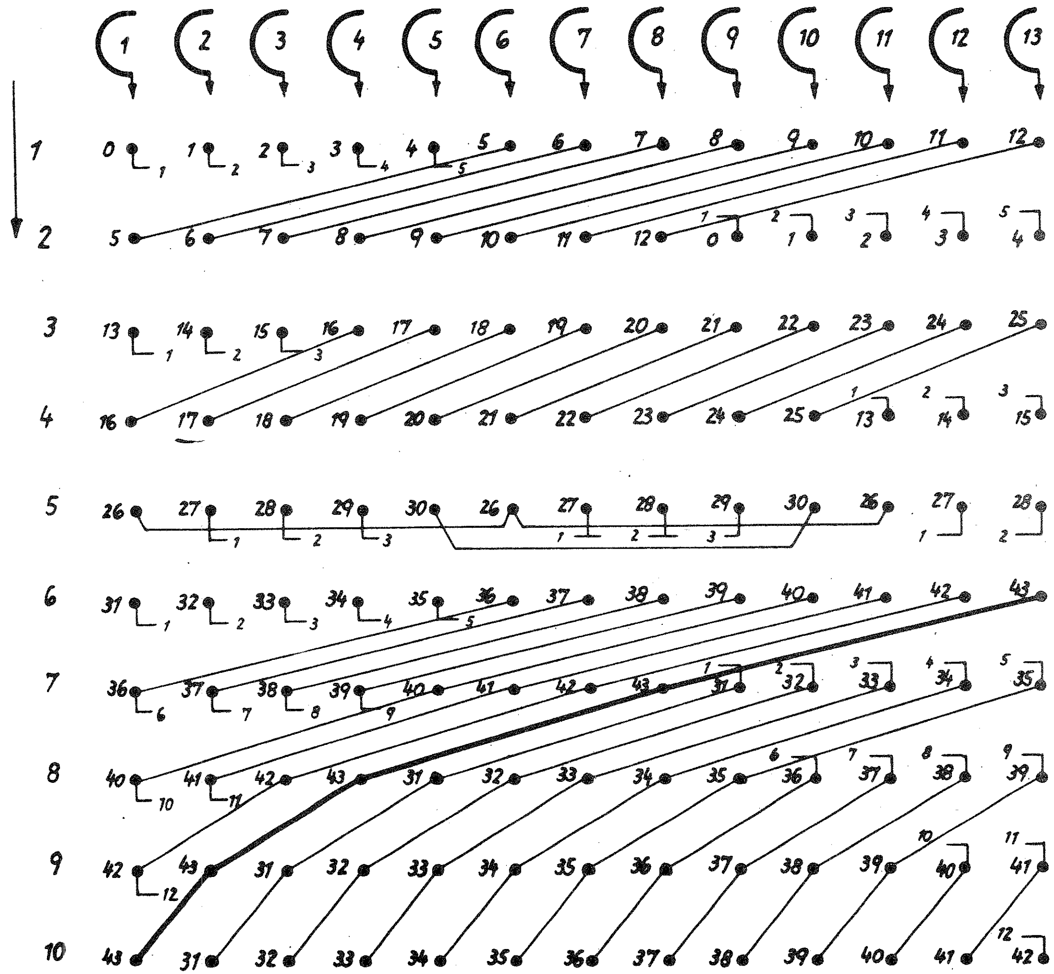
Grading No. 5

$N=40$ $k=6$

$m=40$ $M=\frac{240}{40}=6$



100
10
1
0,1
20
30
40
50
B_k %
N=44
k=10
Test
New Formula
Deutsche Bundespost
Grading No. 6
A(Erl)
fig.4a
Eine Achse logar. geteilt von 1 bis 1000, Einheit 90 mm
Nr. 373 ½ A 4 P



Grading No 6.

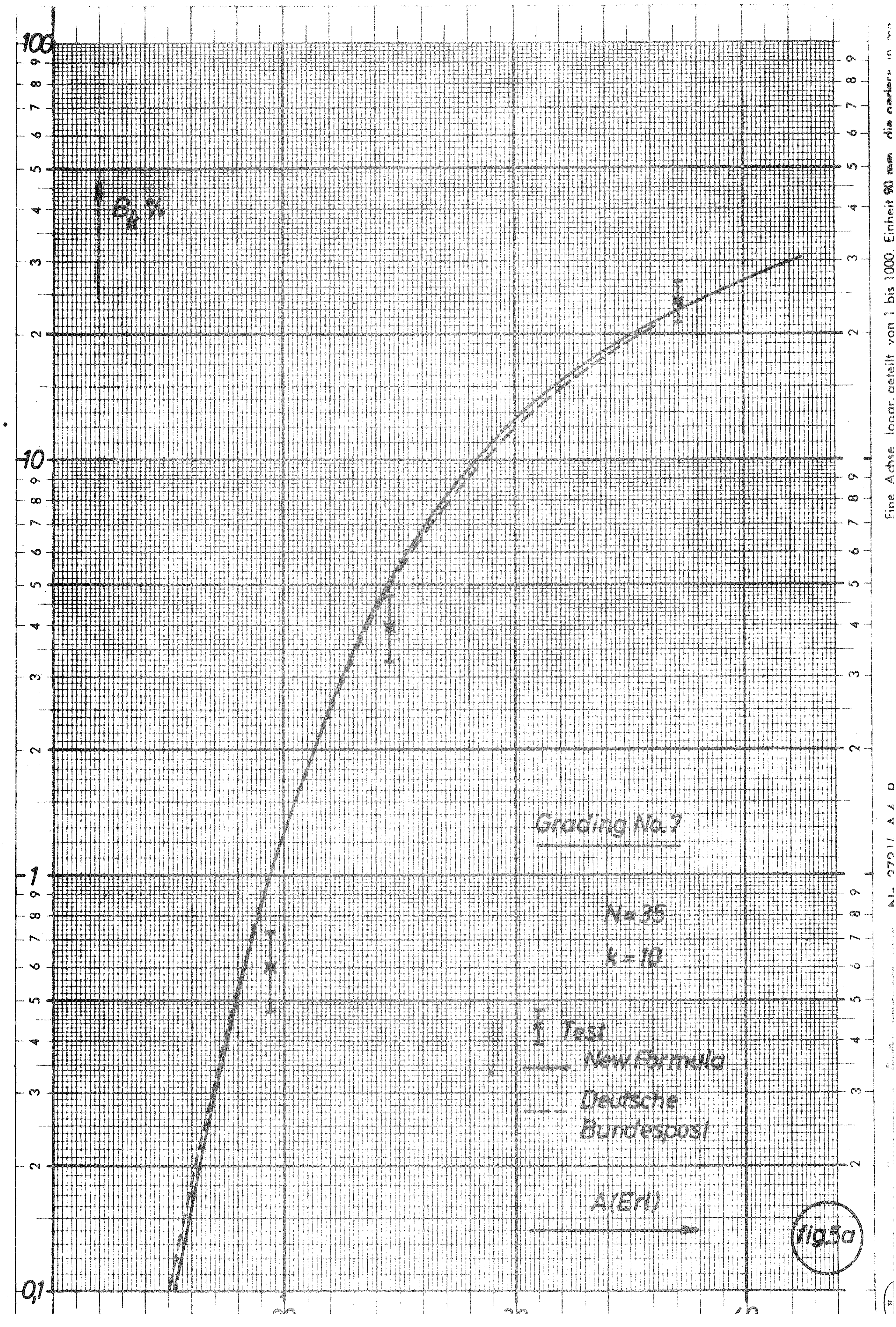
$N = 44$

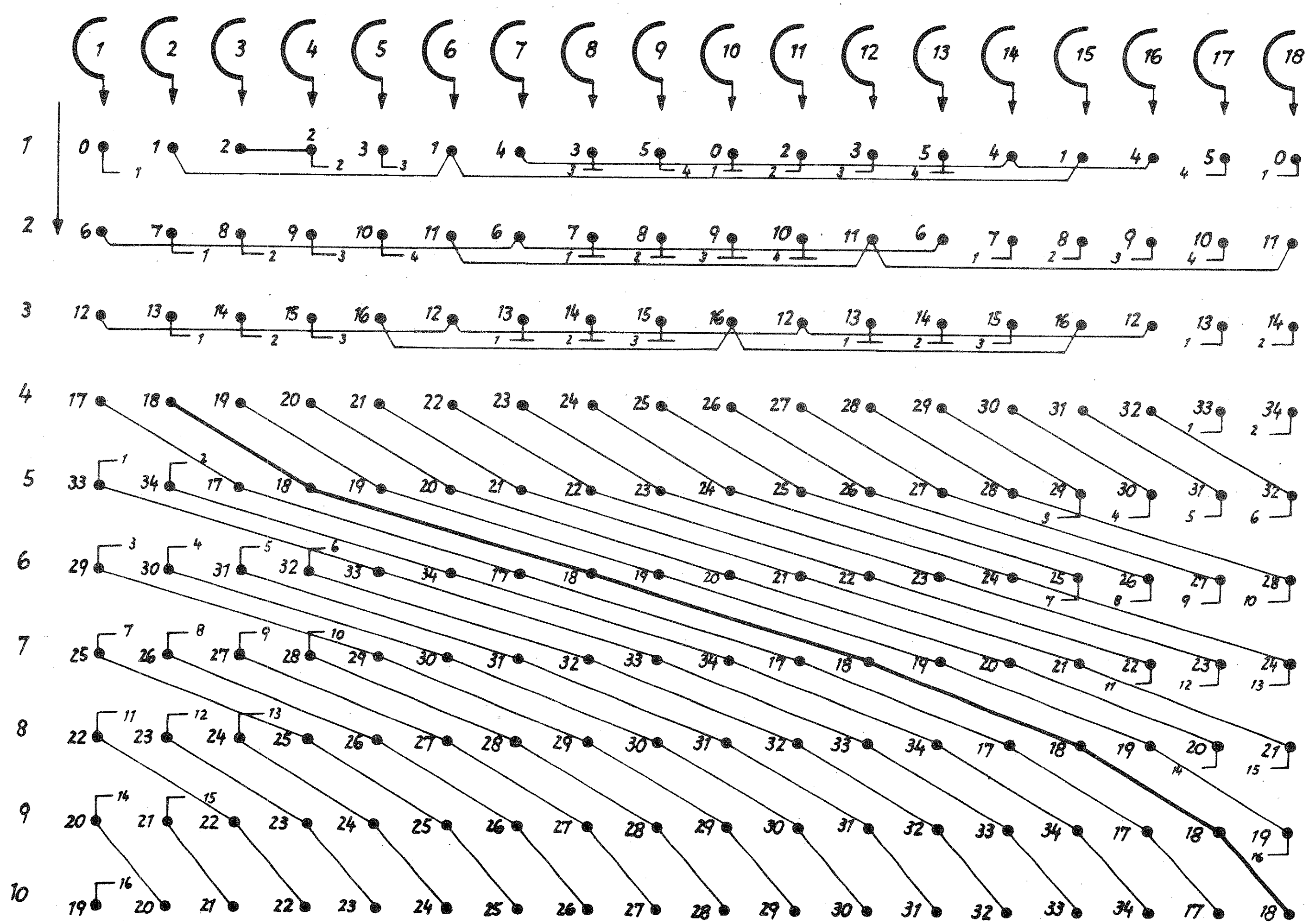
$k = 10$

$m = 13$

$M = \frac{130}{44} \approx 3$

fig.4 b

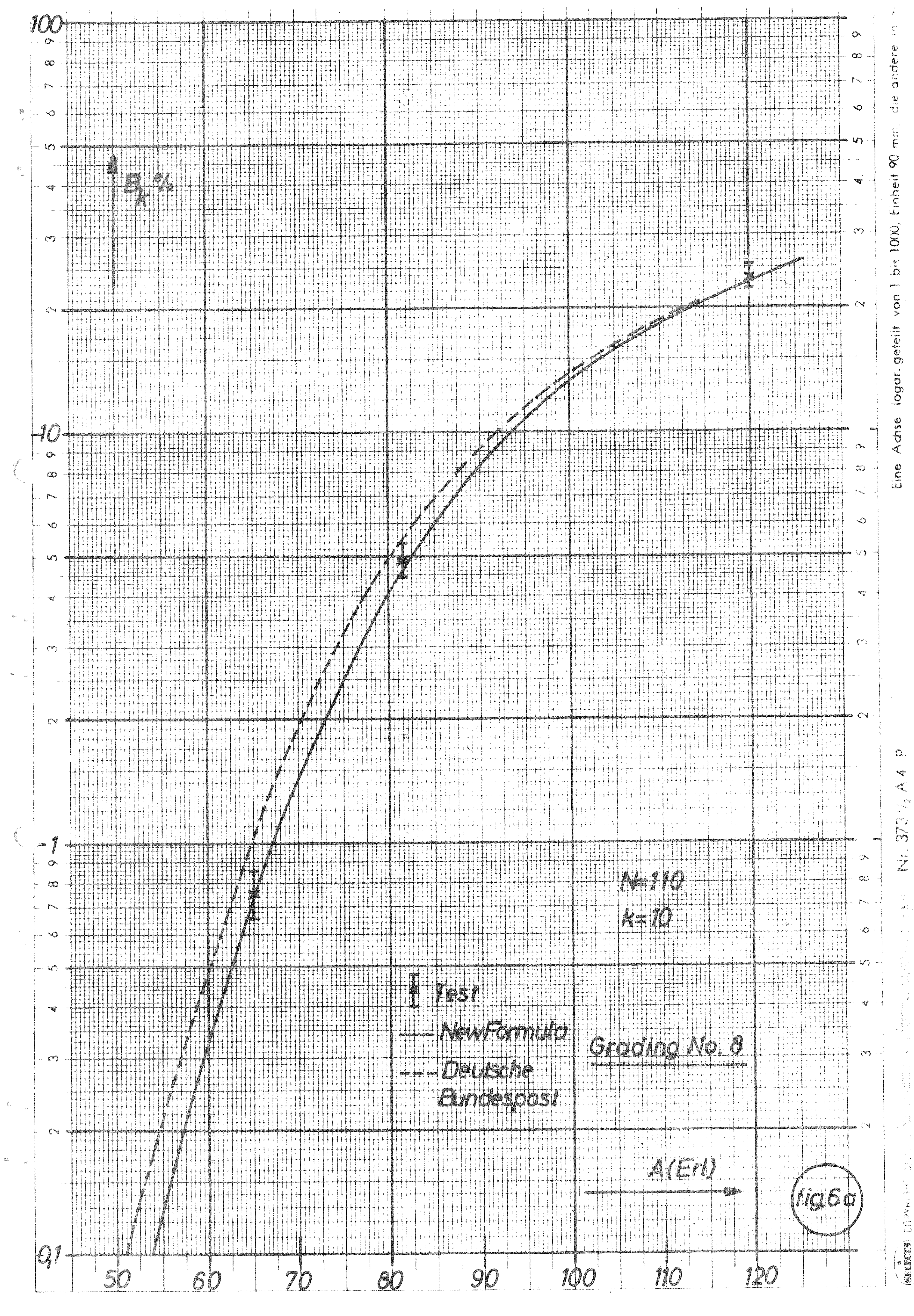

100
10
1
0,1
B_k %
Grading No. 7
N = 35
k = 10
Test
New Formula
Deutsche Bundespost
A(Erl)
fig 5a
Eine Achse logar. geteilt von 1 bis 1000. Einheit 90 mm



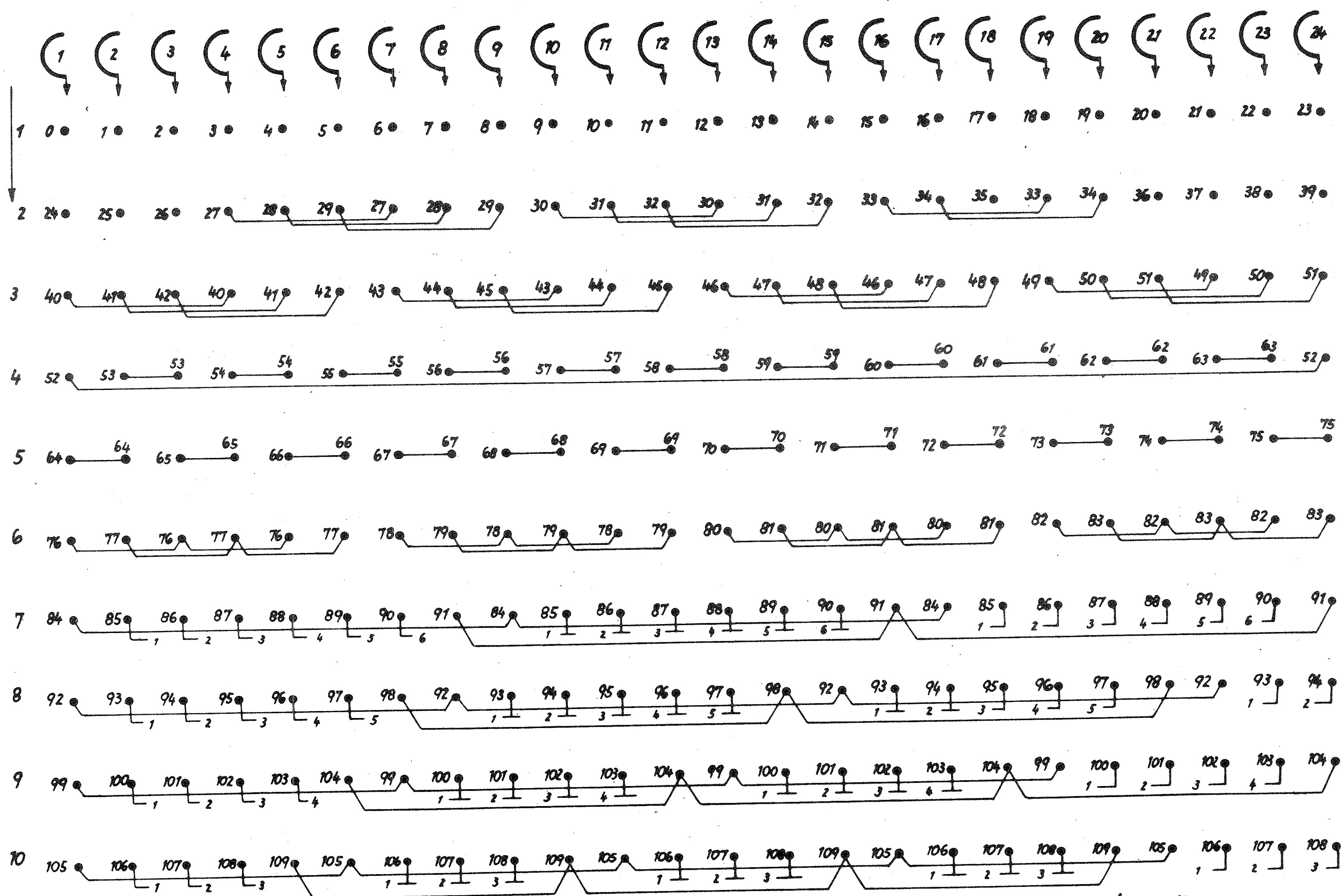
Grading No. 7.

$N = 35$ lines
$k = 10$
$m = 18$
$M = \frac{m \cdot k}{N} \approx 5{,}1$

fig. 5b



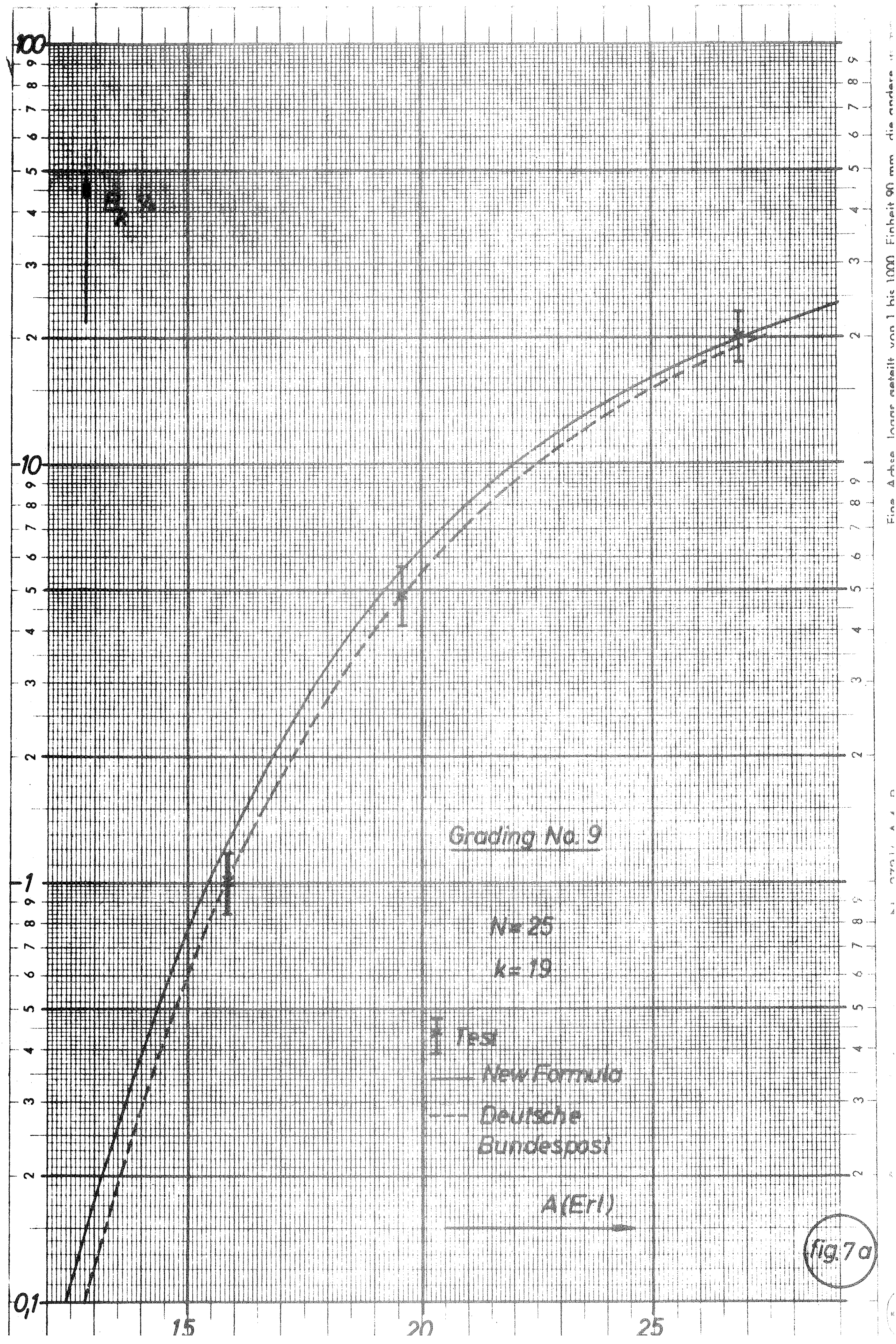
B_k %
N=110
k=10
Test
New Formula
Deutsche Bundespost
Grading No. 8
A(Erl)
fig.6a
100
10
1
0,1
50
60
70
80
90
100
110
120
Eine Achse logar. geteilt von 1 bis 1000, Einheit 90 mm, die andere in
Nr. 373 1/2 A4 P



Grading No 8.

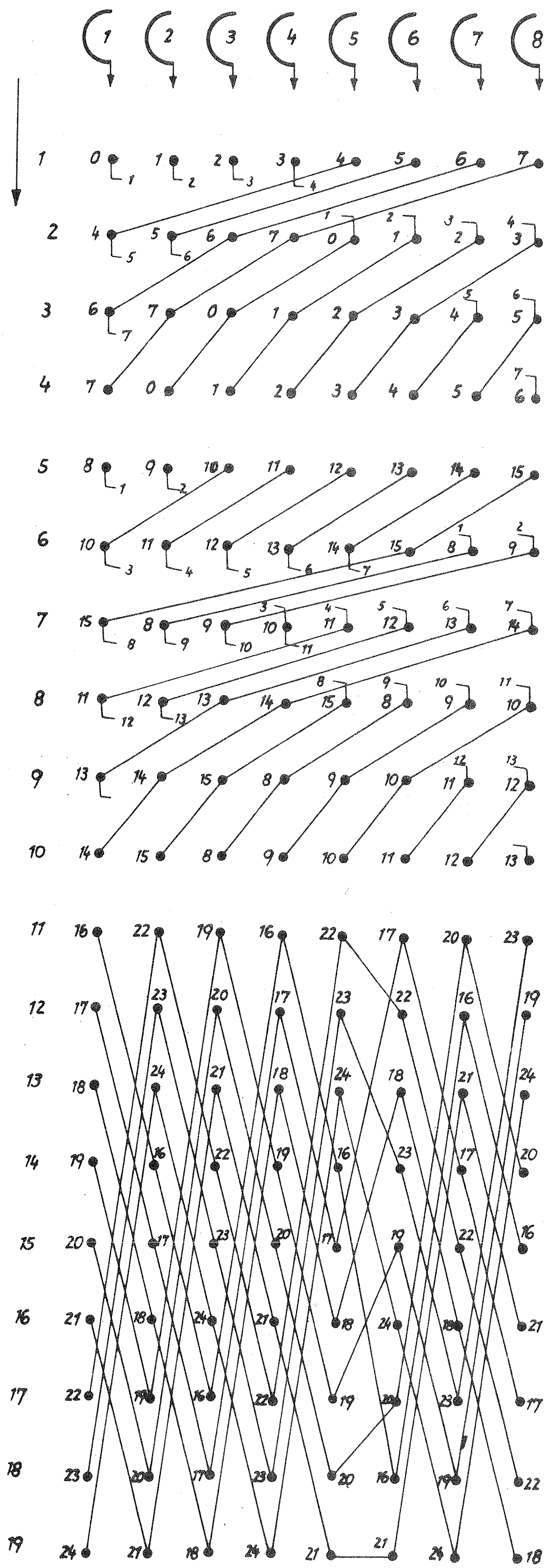
$N = 110$, $m = 24$ | $k = 10$, $M = \frac{240}{110} \approx 2{,}2$

fig6b



B_k %
Grading No. 9
N = 25
k = 19
Test
New Formula
Deutsche Bundespost
A(Erl)
100
10
1
0,1
15
20
25
Eine Achse logar. geteilt von 1 bis 1000, Einheit 90 mm, die andere

fig. 7 a

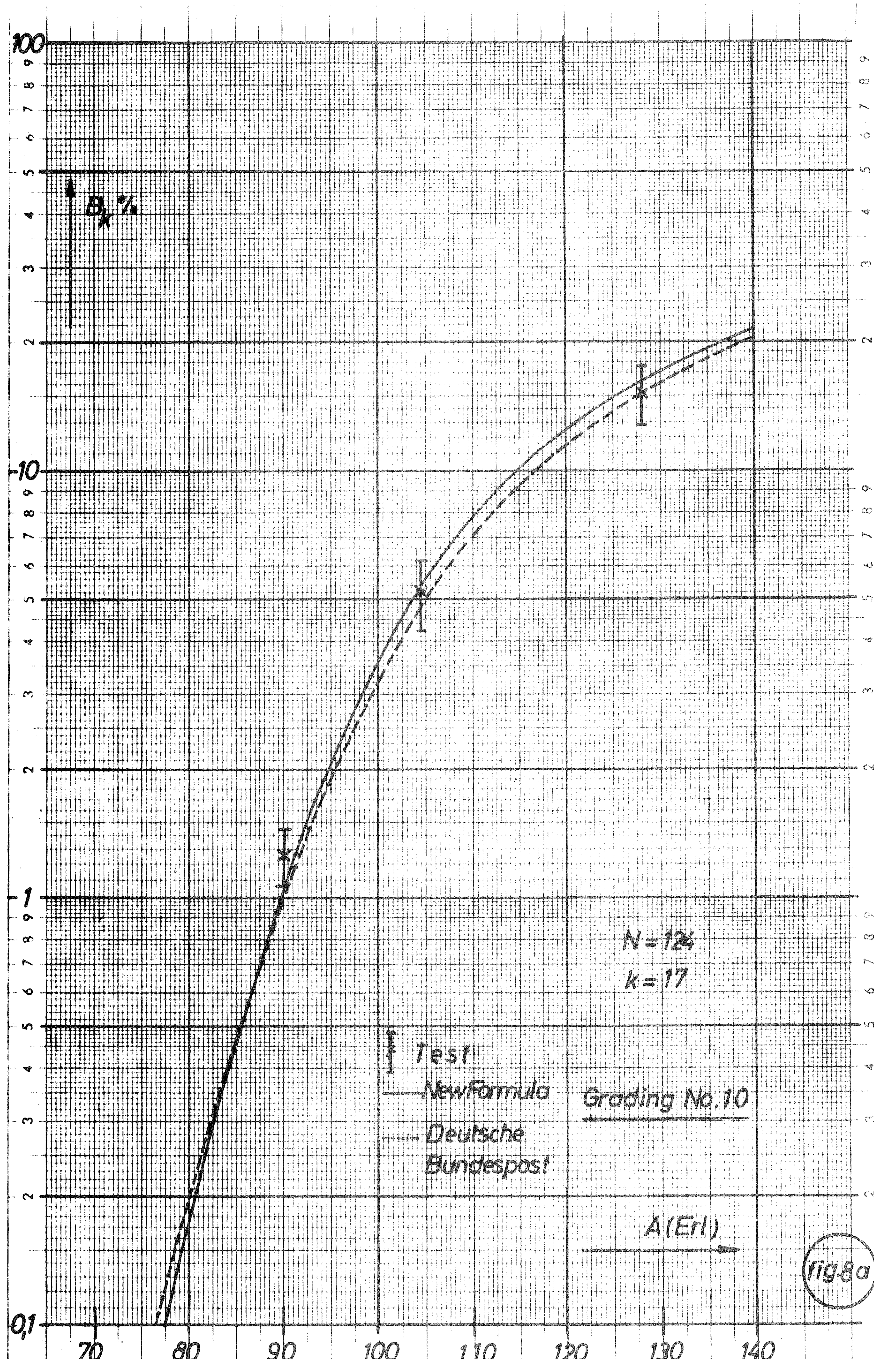


Grading No. 9

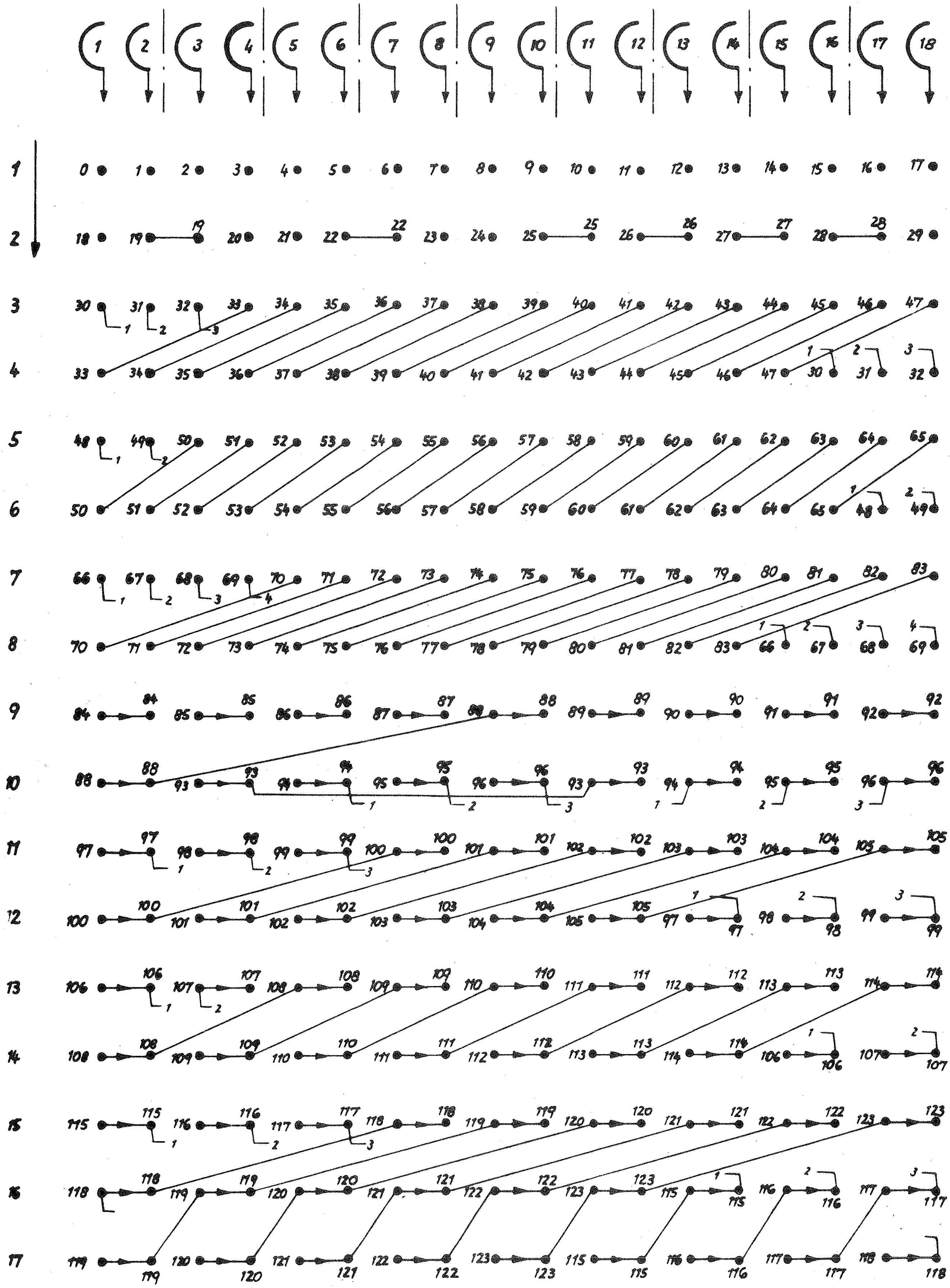
$N=25$
$k=19$

$m=8$
$M=\frac{152}{25}\approx 6$

fig 7b



100
10
1
0,1
B_k %
N = 124
k = 17
Test
New Formula
Deutsche Bundespost
Grading No. 10
A (Erl)
70
80
90
100
110
120
130
140
fig. 8a

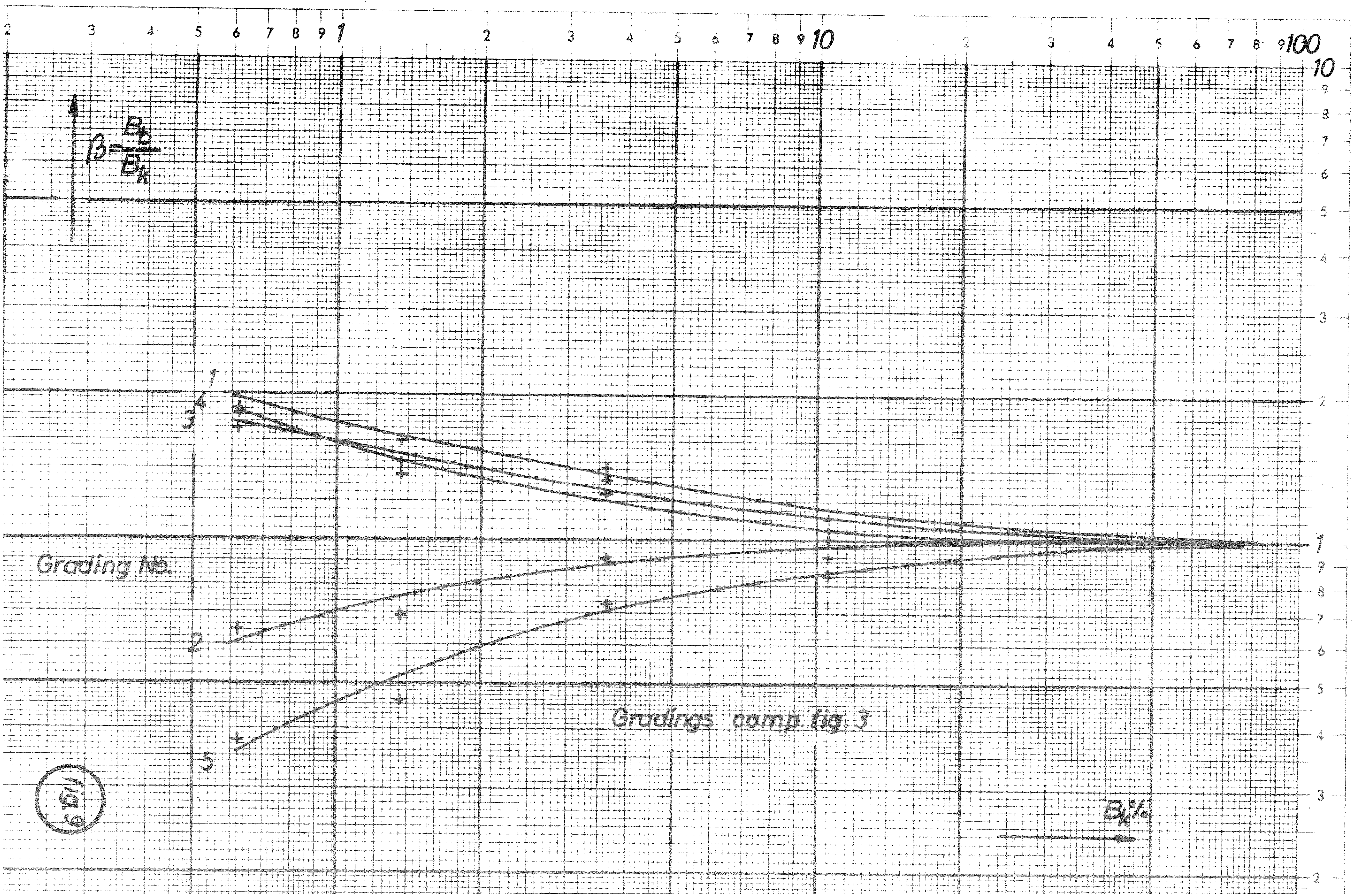


Grading No 10

$N = 124$ | $m = 18$

$k = 17$ | $M = \frac{306}{124} \approx 2{,}5$

fig.8b



$\beta = \frac{B_b}{B_k}$
Grading No.
1
4
3
2
5
Gradings comp. fig. 3
B_k %
fig. 9

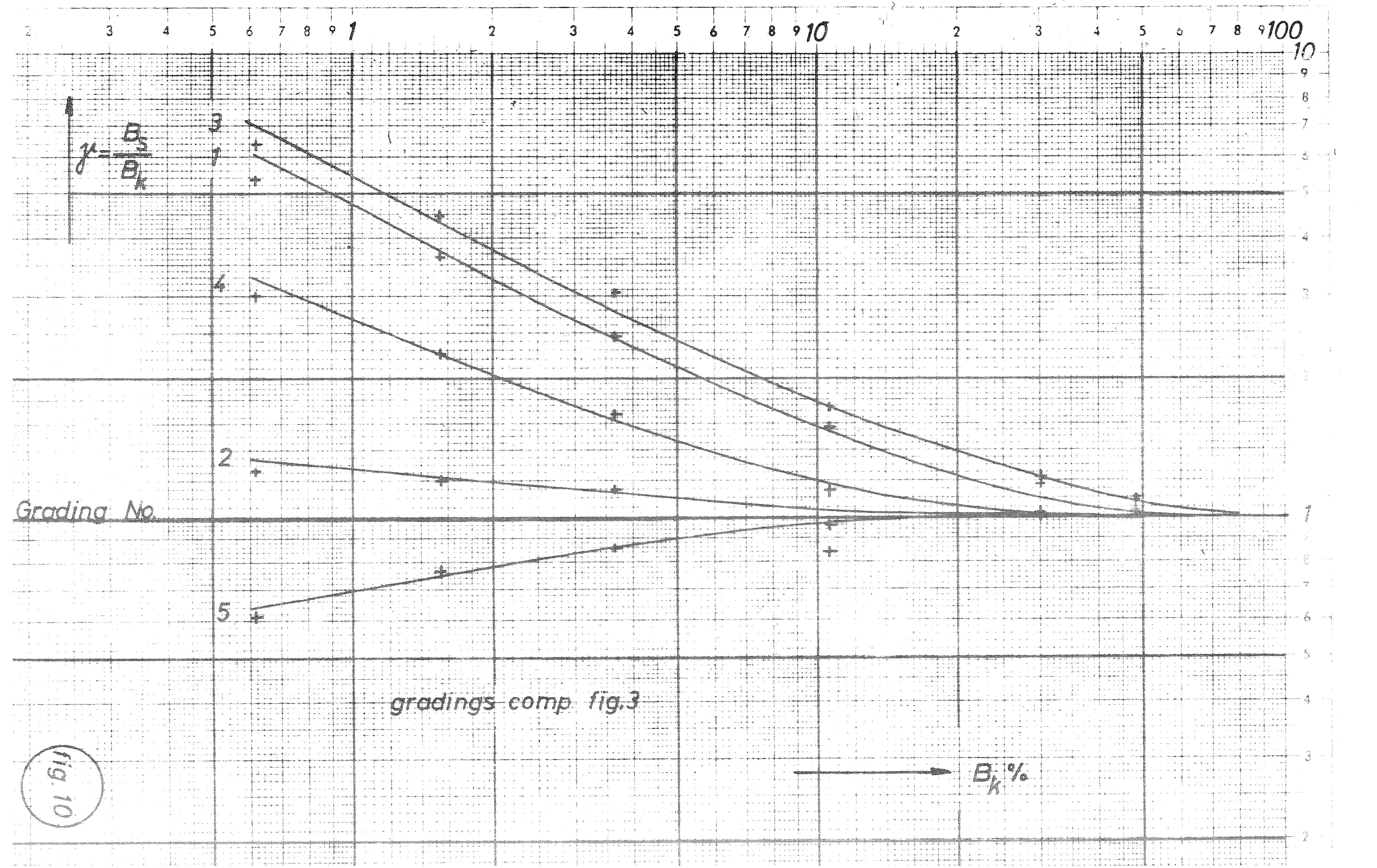


fig. 10