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History and Development of Grading Theory

by Alfred Lotze

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by ALFRED LOTZE *

This report gives a systematic survey of the approximate methods for the calculation of gradings.

The various calculation methods for gradings in loss systems are classified as follows:

- Not truncated distributions.
- Interpolation between boundary values.
- Step by step calculation for sequential hunting and for PCT 1.
- Statistical equilibrium combined with passage probability.
- Presumed truncated distributions combined with passage probability.
- Alternate routing with limited access.

Finally gradings in delay systems are considered.

Entwicklung und derzeitiger Stand der Berechnungsverfahren für Mischungen

Diese Arbeit gibt einen systematischen Überblick über die Methoden zur approximativen Berechnung von Mischungen.

Die zahlreichen Berechnungsmethoden für Mischungen in Verlustsystemen werden wie folgt klassifiziert:

- Verteilungsfunktionen, die unabhängig von der Bündelgröße sind.
- Interpolation zwischen Grenzwerten.
- Schrittweise Berechnung für geordnetes Absuchen und Zufallsverkehr 1. Art.
- Statistisches Gleichgewicht in Verbindung mit der „Durchlaßwahrscheinlichkeit".
- Annahme von Verteilungen, welche die Bündelgröße berücksichtigen, in Verbindung mit der Durchlaßwahrscheinlichkeit.
- Alternative Leitweglenkung mit unvollkommener Erreichbarkeit.

Zum Schluß werden Mischungen in Wartesystemen betrachtet.

1. Introduction

This study gives a concise survey of the development of loss calculation methods for gradings within the past 60 years. Some of the older formulas seem to be rather unpretentious from the present point of view. However, simplicity of formulas and easy evaluation were remarkable advantages in the times before digital computers became the daily tool of traffic engineers — and are often advantages still today. Let us not forget that even A. K. ERLANG himself has published simple approximation formulas derived from his own exact solutions because of the difficulties of evaluation in his time.

Comparing the results of many simple approximations with later and much more sophisticated ones we can sometimes ascertain only so small differences in the range of *small* losses ($B \lesssim 0.005$) that they are not of important practical meaning.

However, modern networks with alternate routing, using high usage groups, require more exact methods of course, but these must always be prepared for easy practical application.

The following fields of pioneer work in grading research must unfortunately be out of the scope of this abridged review.

- Firstly, this is the history of artificial traffic tests, from the first manually handled throw-downs forward to artificial traffic machines (the first machine being published as early as 1928 by ELLIMAN and FRASER) up to the daily use of digital computers for traffic simulation, which has, without any doubt, initiated a new epoch in traffic theory.
- Secondly the widespread and valuable investigations of many authors searching systematically — by means of traffic measurements — for the best possible types of gradings with respect to various applications and different hunting methods.

2. Abbreviations

For the comparison of the various methods of loss calculation it will be useful to apply uniform abbreviations as follows:

g	number of incoming groups or subgroups, respectively,
k	availability (accessibility) per selector multiple,
n	number of trunks (lines) in the outgoing group (route),
m	interconnection number,
$H = gk/n$	average (or uniform) interconnection number of a grading (grading ratio),
A	offered traffic,
Y	carried traffic,
R	overflow traffic (non-random rest of traffic offered),
V, D	variance V or variance coefficient $D = (V - R)$ of overflow traffic R,
x	instantaneous number of existing occupations (calls) in a group, subgroup etc.,
$p(x)$, $w(x)$	probabilities of a state $\{x\}$,
λ	call rate in case of PCT 1,
α	call rate per idle source in case of PCT 2,
E, E_k, E_0	time congestion probabilities,
B, B_k, B_0	call congestion probabilities,
$u(x)$	passage probability in the state $\{x\}$ of a trunk group (expectation value),
$c(x)$	blocking probability in the state $\{x\}$ of a trunk group (expectation value),
q	number of traffic sources.

In the following two types of traffic will be considered:

PCT 1 (Pure Chance Traffic of Type 1). An *infinite* number of sources produces the offered traffic with the mean value A. The total call rate λ is constant and independent of the number of busy sources.

PCT 2 (Pure Chance Traffic of Type 2). A *finite* number of sources produces the offered traffic A. Each idle source has the constant call rate α.

In both cases, the sources are supposed to be independent from each other. Idle sources start calls at random. This implies a negative exponential distribution of idle times of each source.

The distribution of holding times is also assumed in both cases to be negative exponential with the mean value t_m (termination rate: $\varepsilon = 1/t_m$).

Therefore, the offered traffic is given by the following equations:

$$\text{PCT 1: } A = \lambda t_m = \lambda/\varepsilon,$$
$$\text{PCT 2: } A = (q - Y)\,\alpha/\varepsilon$$

where Y is the traffic carried on the trunk group.

3. General Remarks on Gradings

3.1. *The introduction of gradings*

With growing amount of automatic telephone traffic it became more and more uneconomic to apply selectors having such a large number k of contacts that each out of n outgoing trunks of a group could be hunted. On the other

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hand, it led to a rather poor efficiency of the trunks if groups were divided into two or more small separate subgroups (Fig. 1). The large number of grading types having been developed till now cannot be discussed here in detail. A very impressive introduction to the different grading types can be found for example in ELLDIN's publication [1].

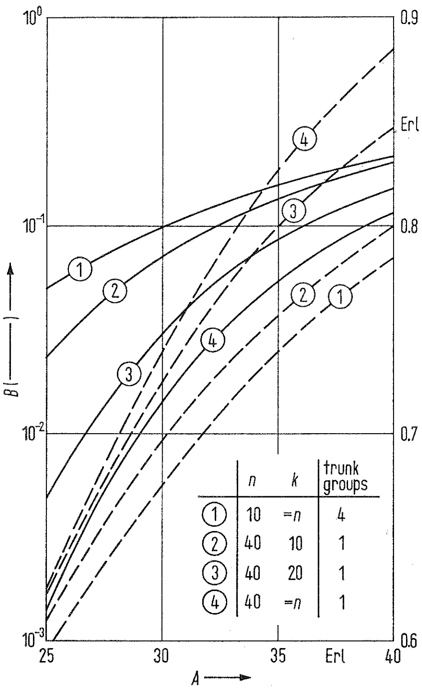

Fig. 1. Call congestion B (———) and carried traffic per trunk Y/n (— — —) as a function of the traffic offered A with $n = 40$ trunks.

Generally spoken, any interconnecting scheme between the outlets of $g \geqq 2$ selector multiples, each multiple consisting of one or more selectors, may be defined as a grading. The selector multiples have k common outlets each (as a rule $k_1 = k_2 = \cdots = k_j = \cdots = k_g$, but not necessarily so), and because of $k < n$ each offered call has a *limited access* to only k out of the n outgoing trunks (lines) of the group.

The interconnections between the different selector multiples form *one* grading if each subgroup influences directly or indirectly the blocking probability of all the other $(g-1)$ subgroups.

The famous first grading patent of E. A. GRAY was filed to the U.S. Patent Office already in July 1907 [2].

3.2. Available formulas for full access

The theory of loss calculation for *gradings* could start from many formulas already in use for full available groups. The best known are counted up here:

PCT 1: 1908 CAMPBELL:

$$n \approx Y + 3.8\sqrt[3]{Y} \quad (B = 0.002).$$

1908 MOLINA and 1918 LELY:

$$B = \mathrm{e}^{-A} \sum_{x=n}^{\infty} \frac{A^x}{x!}.$$

1913 CHRISTENSEN (and ERLANG):

$$B = \mathrm{e}^{-A} \sum_{x=n}^{\infty} \frac{A^x}{x!} \to n \approx Y + c\sqrt[2]{Y}$$

$$(B = 0.001 \to c = 3.3).$$

1917 ERLANG (exact solution):

$$B = \frac{A^n}{n!} \Big/ \sum_{i=0}^{n} \frac{A^i}{i!} = E_{1,n}(A).$$

PCT 2: 1916 MILON: $B = (Y/n)^n$.

1907 MOLINA and 1918 LELY:

$$B = \sum_{x=n}^{q-1} \binom{q-1}{x} t^x (1-t)^{q-1-x}; \quad t = \frac{A}{q}.$$

1918 ENGSET and 1920 O'DELL:

$$B = \frac{\binom{q}{n}\alpha^n}{\sum_{i=0}^{n}\binom{q}{i}\alpha^i} \, \frac{q-n}{q-A}; \quad \alpha = \frac{A}{q-A}.$$

1918 ERLANG, 1923 MARTIN and 1928 FRY (exact solution):

$$B = \frac{\binom{q}{n}\alpha^n}{\sum_{i=0}^{n}\binom{q}{i}\alpha^i} \, \frac{q-n}{q-Y}; \quad \alpha = \frac{A}{q-Y}.$$

3.3. Types of grading formulas

This survey will proceed the following way. It will not be the chronological way in any case, but is appropriate to handle the various methods from simple to more complicated ones:

4.1. Not truncated distributions.
4.2. Interpolation between boundary values.
4.3. Step by step calculation for sequential hunting and PCT 1.
4.4. Statistical equilibrium combined with passage probability.
4.5. Presumed truncated distributions combined with passage probability.
4.6. Alternate routing in case of groups with limited access.

4. Loss Formulas for Gradings

4.1. Not truncated distributions

All methods described here are based on the assumption "no holes in the multiple". From this follows that all states $\{0 \leqq x \leqq n\}$ are assumed to exist merely on the considered actual n trunks of a group.

4.1.1. E. C. MOLINA's formula No. 1 for PCT 1 (1921) [3]

MOLINA continued former pioneer investigations of M. RORTY (1905). MOLINA made the following assumptions:

No. 1: The carried traffic be equally distributed among all n outgoing lines by appropriate grading and hunting methods. Then, for "x trunks busy", each out of $\binom{n}{x}$ patterns is equally probable.

No. 2: The probability of state within the finite number of n outgoing lines be approximately a Poisson distribution (cf. Section 3.2):

$$p(x) = \mathrm{e}^{-A} \frac{A^x}{x!}. \tag{1}$$

From assumption No. 1 the blocking probability $c(x)$ can be derived that a call occuring during the state $\{x\}$ cannot find any idle trunk out of these k trunks which can be hunted by its selector group:

$$c(x) = \binom{x}{k} \Big/ \binom{n}{k}. \tag{2}$$

The passage probability is

$$u(x) = 1 - c(x). \tag{3}$$

From eqs. (1), (2), and (3) one gets after some transformations the expectation of call congestion (MOLINA's formula No. 1), written in the following terms being appropriate for the evaluation:

$$B_{k,n}(A) = \tag{4}$$

$$= A^k \frac{(n-k)!}{n!} \left(1 - \mathrm{e}^{-A} \sum_{x=n-k}^{\infty} \frac{A^x}{x!}\right) + \mathrm{e}^{-A} \sum_{x=n}^{\infty} \frac{A^x}{x!}.$$

From eq. (4) follows the carried traffic

$$Y = A[1 - B_{k,n}(A)]. \tag{5}$$

4.1.2. M. MERKER's loss formula 1924 [4]

MERKER considers a grading with sequential hunting having only two subgroups, each with k_1 individual outlets and k_2 common outlets. Therefore the selectors have the availability

$$k = k_1 + k_2. \tag{6}$$

Congestion arises (approximately)

a) if one selector group has $(k_1 + j)$ and the other one $(k - j)$ busy outlets.

b) if at least all $n = (2k_1 + k_2) = (k_1 + k)$ actual outlets are busy, eventually plus further fictitious outlets (No. $(n+1)$, $(n+2)$, ..., assumption "no holes in the multiple").

The traffic offered to each subgroup be A_s, so the total offered traffic is $A = 2A_s$.

Therewith one obtains two shares of the total loss B:

$$B_a = \mathrm{e}^{-A} \sum_{j=0}^{k-k_1} \frac{A_s^n}{(k_1+j)!\,(k-j)!}, \quad B_b = \mathrm{e}^{-A} \sum_{x=n+1}^{\infty} \frac{A^x}{x!}. \tag{7), (8}$$

Finally

$$B = B_a + B_b. \tag{9}$$

4.1.3. E. C. MOLINA's loss formula No. 2 (1931) [5]

MOLINA investigates the same type of sequentially hunted gradings as MERKER in Section 4.1.2, but for the generalized case of $g \geqq 2$ selector groups (see Fig. 2).

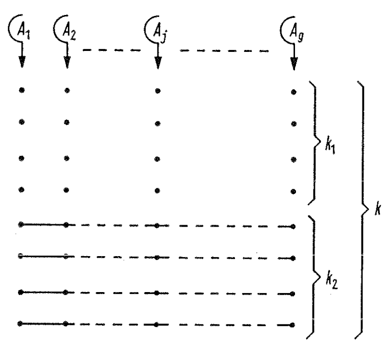

Fig. 2. Type of grading investigated by MOLINA.

He obtains two shares of the total loss B:

a) If in one particular selector group No. j all available $k_1 + k_2 = k$ outlets are busy, caused by its *own* offered traffic A_j (type PCT 1), the share of call congestion for this case becomes

$$B_1 = \mathrm{e}^{-A_j} \sum_{x=k}^{\infty} \frac{A_j^x}{x!}. \tag{10}$$

b) There may exist exactly $(k_1 + r)$ calls originated from the traffic A_j offered to the considered subgroup No. j, where $r \geqq 0$ be prescribed. The corresponding probability is

$$p_1(k_1 + r) = \mathrm{e}^{-A_j} \frac{A_j^{k_1+r}}{(k_1+r)!}. \tag{11}$$

Moreover, at least $(k_2 - r)$ calls may exist on the remaining $(k_2 - r)$ common outlets plus further (fictitious) commons ("no holes in the multiple" assumed). Those $(k_2 - r)$ calls are allowed to originate from any s out of the $(g-1)$ other selector groups.

By means of a sophisticated combinatorial method MOLINA derives the formula for the corresponding ability

$$p_{(g-1)}\{\geqq (k_2 - r)\}$$

which implies all $(g-1)$ other selector groups. Therewith he obtains

$$B_2 = \sum_{r=0}^{k_2-1} p_1(k_1 + r)\, p_{(g-1)}\{\geqq (k_2 - r)\}. \tag{12}$$

Finally

$$B = B_1 + B_2. \tag{13}$$

4.1.4. A. K. ERLANG's approximate formulas [6]

For easy evaluation, ERLANG has derived two simple approximations from his Interconnection Formula (1920). For n, $A \gg k$ and small values of B:

$$B = \left(\frac{A}{n}\right)^k \approx \left(\frac{Y}{n}\right)^k \tag{14}$$

and for $A \ll 1$:

$$B = A^k \frac{(n-k)!}{n!} = \mathrm{e}^{-A} \frac{A^n}{n!} \Big/ \mathrm{e}^{-A} \frac{A^{n-k}}{(n-k)!}. \tag{15}$$

4.2. Interpolation between boundary values

4.2.1. G. F. O'DELL's Method (1927) [7]

The basic idea of O'DELL starts from the *prescribed* call congestion B in case of PCT 1 and the corresponding admissible traffic A_0 offered to a full available group of k lines only, where k means the accessibility of the considered group with $n > k$ trunks. Therewith one obtains the carried traffic on k trunks only

$$Y_0 = [1 - E_{1,k}(A_0)]\, A_0 \tag{16}$$

with $E_{1,k}(A_0) = B$. Then Y_0/k stands for the lower bound of admissible carried traffic per trunk. (The original publication uses A_0 for very small losses. S. A. KARLSSON suggested in [8] the use of Y_0 for the application to higher values of loss.)

With increasing number of lines n and constant accessibility k one obtains the lower bound loss formula according to A. K. ERLANG's approximation for the interconnection formula (see eq. (14)):

$$B = (Y/n)^k. \tag{17}$$

Therewith the upper bound of admissible carried traffic per trunk becomes

$$Y/n = B^{1/k}. \tag{18}$$

By means of eqs. (16), (17), and (18) O'DELL interpolates between lower and upper limit of load per line and writes

$$A = \frac{Y}{1-B} = \left[C B^{1/k} + (1-C)\frac{Y_0}{k}\right] \frac{n-k}{1-B} + A_0. \tag{19}$$

The interpolation factor C was determined by O'DELL from measurements, using straight (socalled O'DELL-) gradings with smoothly progressing interconnection number and sequential hunting:

$$C = 0.53 \ldots B \approx 0.002.$$

EINARSSON, HÅKANSSON, LUNDGREN and TÅNGE thoroughly investigated different kinds of improved gradings using skipped interconnections [9]. They found values of C up to ≈ 0.9. For smoothed traffic O'DELL recommends $C = 1$.

4.2.2. The method of Z. POPOVIČ [10]

POPOVIČ starts — like O'DELL — from a given grading (particularily from cyclic gradings) with n outgoing trunks and accessibility k. The call congestion B is prescribed. Random traffic of the type PCT 1 is assumed.

In opposition to O'DELL the interpolation factor is determined by combinatorial methods for the individual grading in consideration.

Firstly, the admissible traffic A_n to a full available group of n trunks is read out of the ERLANG-tables for the prescribed loss $B = E_{1,n}(A_n)$.

Secondly, the traffic A_k offered to a full available group of k lines is determined similarly ($B = E_{1,k}(A_k)$). For n/k groups having k trunks each a total offered traffic $A_r = (n/k)\, A_k$ would be admissible.

The wanted offered traffic A_g to the considered grading becomes according to POPOVIČ

$$A_g = A_r + (A_n - A_r)\, f\,. \tag{20}$$

The factor f represents the quality of the grading. It is calculated by means of combinatorial formulas, which regard some individual properties of the grading. Extensive investigations into this method have been made by M. HUBER and M. GLAUNER [11].

4.3. Step by step calculation for sequential hunting and PCT 1

4.3.1. Calculation without regard to the overflow variance

The simplest approximate calculation "step by step" can be done by reading out the overflow traffics behind all individual or common outlets of the first hunting position from ERLANG's $E_{1,n}$-table. The appropriate partial overflows are added and are used as the offered *random* traffic to individuals or commons of the next hunting position and so forth.

This method yields of course too optimistic values of loss, because the non-randomness of overflow traffic is neglected.

4.3.2. G. S. BERKELEY's one-parameter method (1949) [12]

BERKELEY implicitly takes into account an approximate variance of overflow. His method may be explained by means of the simple example in Fig. 3.

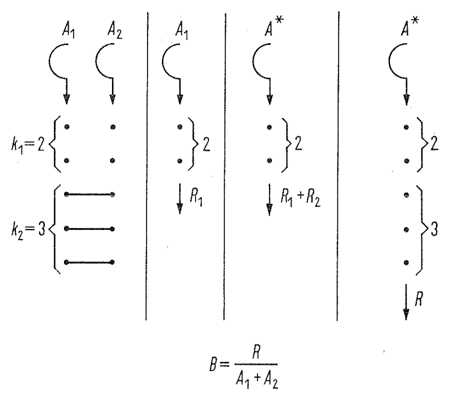

Fig. 3. BERKELEY's loss calculation.

The overflow traffics R_1 and R_2 behind k_1 steps (here $k_1 = 2$) are calculated by means of ERLANG's $E_{1,n}$-tables, where
$$R_1 = A_1 E_{k_1}(A_1) \quad \text{and} \quad R_2 = A_2 E_{k_1}(A_2)\,.$$

Then an auxiliary offered traffic A_0 has to be chosen in the ERLANG-tables such that for k_1 trunks

$$A_0\, E_{k_1}(A_0) = R_1 + R_2\,. \tag{21}$$

Now the next steps (here $k_2 = 3$) which are hunted by the sum of the overflow traffics $R_1 + R_2$ are added and one obtains
$$R = A_0\, E_{k_1+k_2}(A_0)\,. \tag{22}$$
R being the approximate value of the actual overflow traffic of the considered grading. Therefore the call congestion is obtained by
$$B = \frac{R}{A_1 + A_2}\,. \tag{23}$$

The method yields good approximate values of B for straight O'DELL gradings without skipping, if the overflowing partial traffics are not too much correlated because of commonly hunted preceding steps. Correlation diminishes the total variance of the different partial overflows. Then the loss calculation tends towards the safe side.

Graphs and tables according to BERKELEY's method as well as artificial traffic tests have been published by R. R. MINA [13].

4.3.3. The two parameter method (equivalent random method)

BERKELEY's approximate regard of the overflow variance is replaced by using overflow traffics with two correct parameters, i.e. mean R and variance V.

This method has been developed and published firstly by R. I. WILKINSON and J. RIORDAN [14] on the 1. ITC 1955 and later on by G. BRETSCHNEIDER [15]. Fundamental theoretical investigations about the overflow problem by VAULOT (1935) [16], KOSTEN (1937) [17], MOLINA (1941), NYQUIST (both see [14]) and GILTAY (1953) [18] lead the way to this method. The exact overflow distribution has been calculated by R. BROCKMEYER already in 1954 [19].

The way of calculation resembles that of BERKELEY. By means of the small grading example in Fig. 3 we can distinguish the considered method and the BERKELEY method.

Firstly, the overflow traffics R_1 and R_2 are read out of tables or graphs *as well as their corresponding values* V_1, V_2. Instead of an auxiliary traffic A_0 only, *another equivalent random traffic* A^* *and a corresponding number* n^* of full available trunks have to be chosen such a way that the overflowing traffic has *both* the mean $R_\mathrm{m} = R_1 + R_2$ and the variance $V_\mathrm{m} = V_1 + V_2$. To these n^* trunks the next $k_2 = 3$ common hunting steps are added and the lost traffic becomes
$$R = A^*\, E_{n^*+k_2}(A^*) \tag{24}$$
and the call congestion is calculated according to eq. (23).

This method yields mostly somewhat larger losses than the BERKELEY method. In many gradings the main reason for this fact is the increasing correlation between different partial overflows with increasing hunting position particularly if skipping is applied. This correlation, which diminishes the overall variance, is not regarded if the correct values V_i are added linearly. The merely approximate regarding of overflow variance by the BERKELEY method tends to variances which are smaller than the actual ones. This inaccuracy mostly results in a certain correction of this "correlation-effect". As with the BERKELEY method, straight gradings without skipping are therefore best suited for the equivalent random method.

The dominating importance of the two parameter equivalent random method does not concern the calculation of gradings but the design of alternate routing systems.

4.4. Statistical equilibrium combined with passage probability

4.4.1. General remarks

In any group, no matter if there is full or limited access to n trunks, $(n+1)$ different states are possible, i.e. 0, or 1, or 2, ..., or x, ..., or $(n-1)$, or n trunks busy. Each state $\{x\}$ is composed of $\binom{n}{x}$ different patterns, each having a certain individual pattern probability $p((x)_\nu)$, from which the total probability of state $\{x\}$ follows with

$$p(x) = \sum_{\nu=1}^{\binom{n}{x}} p((x)_\nu). \tag{25}$$

In full available groups it makes no sense to distinguish the various pattern probabilities, if merely the time- or call congestion have to be calculated. However, in gradings with limited access $k < n$ all patterns of the states $\{x \geqq k\}$ can (but must not) cause congestion for one ore more out of all g selector groups, depending on their individual topological busy positions among the crosspoints of the grading. From this fact follows that, strictly spoken, in the most general case (without simplification by symmetries) the probabilities of all individual patterns

$$\sum_{x=0}^{n} \binom{n}{x} = 2^n \tag{26}$$

should be calculated. This leads to the strictly exact ways of solutions by means of huge systems of linear equations having up to 2^n unknowns (see [20]—[22]).

The conditional probability $w((x)_\nu | x)$ that within the existing state $\{x\}$ the pattern $(x)_\nu$ exists can be calculated with these exact methods by means of socalled *transition probabilities*. These probabilities tell us from which certain patterns $(x-1)_\nu$ a certain pattern $(x)_\nu$ can be *born* caused by a successful call; and on the other hand, which group of certain patterns $(x-1)_\nu$ can arise when our considered pattern $(x)_\nu$ *dies* because any one of these x calls is terminated.

Let us now reflect on a method which could simplify the calculation without giving up the basic principle of Birth and Death Process, used for the strictly exact calculations.

Because of the generally assumed time invariance of the considered stochastic process of traffic flow, each state $\{x\}$ as a whole (including all its $\binom{n}{x}$ patterns) will on the average be born as often as it will die. In other words, we make use of the condition applied for the first time by A. K. Erlang and named the "*Statistical Equilibrium*" [6]:

The mathematical expectation, that during the existing state $\{x\}$ an occuring call is successful and that the state $\{x\}$ "*dies*" in upward direction, so that any one of the possible patterns $(x+1)_\nu$ is born, be named the *State-Passage Probability* $u(x)$. Obviously $u(x)$ can be written

$$u(x) = \sum_{\nu=1}^{\binom{n}{x}} w((x)_\nu | x)\, u((x)_\nu). \tag{27}$$

$u((x)_\nu)$ being defined by

$$\frac{\text{non-blocked selector groups in case of } (x)_\nu}{\text{all } g \text{ selector groups}}. \tag{28}$$

Assuming moreover a negativ exponential distribution of the holding time and with the mean holding time h being unity, the probability density $d(x+1)$ for the death of any existing pattern $(x+1)_\nu$ in downward direction, that is to say "for the birth of any pattern $(x)_\nu$", becomes

$$d(x+1) = (x+1)\frac{1}{h} = x+1. \tag{29}$$

For Poissonian offered traffic (PCT 1) with the traffic intensity A, we can — by means of eqs. (27) and (29) — write the recurrence formula for the statistical equilibrium:

$$\underbrace{A\,u(x-1)\,p(x-1) + (x+1)\,p(x+1)}_{\text{birth of the state } \{x\}} = \underbrace{A\,u(x)\,p(x) + x\,p(x)}_{\text{death of the state } \{x\}}. \tag{30}$$

Because of the condition $\sum_{x=0}^{n} p(x) = 1$ the above recurrence formula yields

$$p(x) = \frac{\dfrac{A^x}{x!}\prod_{z=0}^{x-1} u(z)}{1 + \sum_{j=1}^{n} \dfrac{A^j}{j!}\prod_{z=0}^{j-1} u(z)}. \tag{31}$$

The complement to the state-passage probability $u(x)$ be named the state-blocking probability

$$c(x) = 1 - u(x) \tag{32}$$

where

$$c(x) = 0 \quad \text{for} \quad x < k.$$

From eqs. (31) and (32) follows the time congestion E, which in the case of PCT 1 equals the call congestion B

$$E_{k,n} = B_{k,n} = \sum_{x=k}^{n} c(x)\,p(x). \tag{33}$$

The difficulty in using this method lies in the sufficiently close approximation of $u(x)$ or $c(x)$, respectively. Let us now consider some of the most interesting applications.

4.4.2. A. K. Erlang's Interconnection Loss Formula (1920) for PCT 1 (EIF) [6]

A. K. Erlang prescribes for his socalled Ideal Erlang Grading

$$g_1 = \binom{n}{k} \quad \text{or} \quad g_2 = \binom{n}{k} k! \tag{34}$$

a) With g_1 selector groups and with balanced offered traffic, each selector group has access to another combination "k out of n trunks". Because of this special grading each out of the possible different $\binom{n}{x}$ states $\{x\}$ blocks exactly $\binom{x}{k}$ out of all $\binom{n}{k}$ selector groups and effects the same state-blocking probability $c(x)$,

$$c(x) = \binom{x}{k} \Big/ \binom{n}{k} = 1 - u(x). \tag{35}$$

Inserting eq. (35) in eq. (31) and (33) we get Erlang's famous Interconnection Formula.

In case of random hunting, all x-patterns become equally probable, therewith the individual call congestions of all g selector groups equal B.

b) Using selectors with home position, the $\binom{n}{x}$ patterns per state $\{x\}$ are not equally probable any more, even in case of balanced offered traffic. Nevertheless, the values $c(x)$ and $u(x)$ remain unchanged.

Therefore, eq. (33) yields the correct overall call congestion B. The call congestions of the individual g selector groups cannot be calculated, because they depend on the unknown pattern probabilities $w((x)_\nu)$, which in their turn are traffic-dependent.

c) Using now $g_2 = \binom{n}{k} k!$ selector groups, the formulas $c(x)$ and $u(x)$ according to eq. (35) do not vary. However, each combination "k out of n trunks" is hunted by the selector groups with home position in each out of $k!$ possible permutations. All $\binom{n}{x}$ patterns of a state $\{x\}$ will then be equally probable again. Therefore eq. (33) yields the correct overall call congestion B as well as the uniform values B for each selector group (balanced traffic assumed).

For practically realized gradings with $g < \binom{n}{k}$ selector groups, mostly somewhat too optimistic values of call congestion B are obtained from the EIF. On the other hand, the EIF does not stand for a lower limit of call congestion B (see [6], p. 118).

4.4.3. H. A. LONGLEY's investigations (1948) [23]

LONGLEY calculates small gradings of different type exactly by solving the 2^n equations required. Inserting all exactly calculated probabilities of state $p(x)$ into the equations of type (30) he obtains the values $u(x)$ and $c(x)$. The terms $u(x)$ are called K-factors in LONGLEY's publication.

In a second step, larger gradings were investigated where the evaluation of 2^n unknowns was impossible. Here the maximum and minimum values of all K-factors were calculated merely for the much easier limits $A \to \infty$ and $A \to 0$.

LONGLEY found that the differences between the two K-series for $A \to \infty$ and $A \to 0$ were rather small. Because of these results a simple approximate formula for $u(x)$ is given, which can be inserted into eqs. (31), (33) for the calculation of loss.

4.4.4. Approximate assumption No. 1: All $\binom{n}{x}$ patterns $(x)_\nu$ of a state $\{x\}$ are equally probable

a) From this assumption No. 1 follows that each selector group will have the same state-congestion probability. Considering one arbitrarily chosen selector group, whose k outlets are busy in the state $\{x\}$, we get for the remaining $(x-k)$ existing calls on $(n-k)$ lines $\binom{n-k}{x-k}$ equally probable patterns. Therefore (as for the EIF see Section 4.2)

$$c(x) = 1 - u(x) = \binom{n-k}{x-k} \Big/ \binom{n}{x} = \binom{x}{k} \Big/ \binom{n}{k}. \quad (36)$$

$p(x)$ and B result from eqs. (31) and (33).

b) If $c(x)$ and $u(x)$ represent sufficiently good expectation values according to eq. (27) though assumption No. 1 does not hold exactly, the overall loss B accords with eq. (33), whereas the individual losses of the g selector groups may vary.

4.4.5. Approximate assumption No. 2:

a) According to Section 4.4.4. b) it suffices to have adequately *exact approximate values for the expectation* $u(x)$ where the explicit single values of the sum in eq. (27) must not be known. If $u(x)$ is close to reality the overall call congestion B according to eq. (33) yields results close to reality, too.
The actual values $w((x)_\nu \mid x)$ and $u((x)_\nu)$ can vary the individual losses of the individual selector groups but not the value B, if $u(x)$ is sufficiently exact.
The approximate function $u(x)$ can, of course, differ from eq. (35) and can be obtained only by measurements or by combinatorial methods with or without respect to some individual properties of the gradings in consideration [24]—[27].

b) J. N. BRIDGFORD's geometric group concept (1961) [26] The state-blocking probability $c(x)$ starts with

$$c(n-1) = 1 - \frac{k}{n}. \quad (37)$$

This starting value (called "p" in [26]) is exact for homogeneous gradings and is also obtained by means of eq. (35). Then the series is continued

$$c(n-2) = p^2, \quad c(n-3) = p^3, \quad \text{etc.} \quad (38)$$

The method has been applied to grading types used by the Australian GPO.

4.4.6. General remarks to gradings having a finite number of sources (PCT 2)

The general remarks in Section 4.4.1 hold also for gradings with offered PCT 2. In this case, however, strictly exact solutions cannot be obtained. The instantaneous traffic intensity offered to each individual selector group depends on that number of its sources which take part in the instantaneous existing pattern $(x)_\nu$. Therefore, the application of "Statistical Equilibrium" can yield approximations only. (Strictly exact solutions are — in the general case — available by means of the solution of 2^n equations of state.) The recurrence formula (for "lost calls cleared") holds:

$$\alpha(q-x+1)\,u(x-1)\,p(x-1) + (x+1)\,p(x+1) = \alpha(q-x)\,p(x)\,u(x) + x\,p(x). \quad (39)$$

Therewith the probability of a state $\{x\}$ becomes

$$p(x) = \frac{\binom{q}{x} \alpha^x \prod_{z=1}^{x-1} u(z)}{1 + \sum_{i=1}^{n} \binom{q}{i} \alpha^i \prod_{z=1}^{i-1} u(z)}. \quad (40)$$

The time congestion is given by

$$E_k(\alpha, n, q) = \sum_{x=k}^{n} c(x)\,p(x) \quad (41)$$

and the call congestion by

$$B_k(\alpha, n, q) = \left[\sum_{x=k}^{n} (q-x)\,p(x)\,c(x)\right] \Big/ q - Y. \quad (42)$$

4.4.7. The method of K. ROHDE and H. STÖRMER [27]

This theory is based on the approximate assumptions "lost calls held" and "no holes in the multiple". The offered traffic is defined by a value

$$A^* = q\,p \quad (43)$$

with q being the number of traffic sources and p being the probability for each out of q independent sources to be busy *under the condition* that an unlimited traffic flow could exist, using $n^* = q$ full available trunks. Starting from these assumptions and using the *actual* number $n < q$ of trunks hunted with limited access $k < n < q$, the principle of statistical equilibrium is applied. The following loss formula (referred to A^*) is obtained:

$$B_k(A^*, n, q) = \sum_{x=k}^{n} c(x)\,p(x)\,\frac{q-x}{q}. \quad (44)$$

The value $c(x)$ for gradings is an approximation, derived from a combinatorial solution for two-stage link systems (the exact derivation under their assumptions leads to eq. (35)). The actually applied approximations yield too pessimistic values of loss for gradings as used in practice [24].

4.4.8. Method No. 1 by D. BOTSCH for gradings carrying external and internal traffic [28]

In gradings used for both-way traffic a certain share of calls which originate from a certain subscriber (or inlet) of the grading will be connected to another subscriber (or inlet) which belongs to the *same* grading. In this case the traffic between two subscribers (inlets) of the same grading runs *twice* through this considered grading. This share of traffic will be called *internal* traffic and needs *two* trunks of the same group for one call. The other share of traffic runs only once through the grading and will be called *external* traffic. N. RØNNBLOM [29] and later on R. FORTET and CH. GRANDJEAN [30] as well as D. BOTSCH [28] solved this problem for full available groups and PCT 1.

For gradings two solutions for PCT 1 and PCT 2 each have been developed by BOTSCH. The solution No. 1 for PCT 1 and PCT 2 is described here, solution No. 2 follows in Section 4.5.

From the assumption of statistical equilibrium and by means of a function $u(x)$ for the expectation value of the passage probabilities follows the recurrence formula for a finite number q of sources:

$$p(x+2) = \frac{q-x-1}{x+2}\,\alpha_{\text{ext}}\, p(x+1)\, u(x+1) + \\ + \frac{q-x}{x+2}\, 2\,\alpha_{\text{int}}\, p(x)\, u(x)\, u(x+1)\,. \tag{45}$$

α_{ext} and α_{int} being the call rates per idle source with respect to the external and internal traffic; the holding time h being unity.

To get time- and call congestion, the values for the probability distribution $p(x)$ obtained from eq. (45) have to be inserted into the following equations:
time congestion:

$$E_k = \sum_{x=k}^{n} p(x)\, c(x) + \frac{\alpha_{\text{int}}}{\alpha_{\text{ext}} + \alpha_{\text{int}}} \sum_{x=k-1}^{n-1} p(x)\, u(x)\, c(x+1)\,, \tag{46}$$

call congestion:

$$B_k = \frac{1}{q-Y}\left[\sum_{x=k}^{n} (q-x)\, p(x)\, c(x) + \right. \\ \left. + \frac{\alpha_{\text{int}}}{\alpha_{\text{ext}} + \alpha_{\text{int}}} \sum_{x=k-1}^{n-1} (q-x)\, p(x)\, u(x)\, c(x+1)\right]. \tag{47}$$

Formulas for Poisson input PCT 1 follow by passing to the limit $q \to \infty$

4.5. Presumed truncated distributions combined with passage probability

4.5.1. The PALM-JACOBAEUS loss formula [31], [32]

This method has firstly been suggested by C. PALM [31], later on transformed to the formula (48) by C. JACOBAEUS [32] who successfully used it also for his famous link-system calculations.

The expectation value of state-blocking probability $c(x)$ is chosen according to eq. (36).

Assuming small losses and offered Poisson traffic A (PCT 1) an Erlang distribution $E_{1,x}(A)$ as in a full available group is presumed approximately.

From this follows for n lines and the accessibility k

$$E_k = B_k = \sum_{x=k}^{n} c(x) \left(\frac{A^x}{x!} \Big/ \sum_{i=0}^{n} \frac{A^i}{i!}\right) = \frac{E_n(A)}{E_{n-k}(A)}\,. \tag{48}$$

This formula yields — in the considered range of small losses ($B \lesssim 1\%$) — and for good progressive gradings with skipping very good results close to reality.

4.5.2. The modified PALM-JACOBAEUS loss formula [35]—[38]

In 1928 TH. C. FRY [33] has calculated the probability $S(>k)$ that a selector without home position needs more than k steps to find an idle trunk within a full available group having n lines. When the offered traffic is A_0, an Erlang distribution with the mean carried traffic Y is produced:

$$Y = A_0\{1 - E_n(A_0)\}\,. \tag{49}$$

Analogously to eq. (48) this leads to the exact formula

$$S(>k) = \frac{E_n(A_0)}{E_{n-k}(A_0)}\,. \tag{50}$$

Not knowing this early solution of TH. C. FRY, the same idea was mentioned by A. JENSEN 1952 [34] in a Danish publication and later on once again (1960) by the author as a useful modification of the PJ-Formula to extend its validity up to high values of loss:

$$B_k = \frac{E_n(A_0)}{E_{n-k}(A_0)}\,. \tag{51}$$

A_0 being the "generating PCT 1" of a full access group and with the prescribed *carried* traffic Y according to eq. (49). From this follows

$$A_{\text{actual}} = Y/(1 - B_k)\,. \tag{52}$$

By a large number of artificial traffic tests with the availabilities $2 \leq k \leq 100$ it could be shown, that eq. (51) is indeed very close to reality for progressive (sequentially hunted) gradings with good skipping and a sufficiently large average interconnection number

$$H \approx \frac{2.1}{\log(k+1)}\,. \tag{53}$$

The MPJ-formula has been tabulated in [37] and [38].

Simplified gradings, however, without sophisticated skipping can often save costs remarkably with respect to the installation of local exchanges as well as to the enlargement of grading groups.

Thorough investigations [39] and [40] have shown that — in the interesting range from at least $B = 0.001$ up to $B = 0.50$ — the simplification of gradings shifts the MPJ-loss function $B_k = f(A, k, n)$ practically parallel to the axis of the actual offered traffic A. The amount of the adapting shifting-value ΔA_1 can be easily calculated by the empirical loss-*independent* formula

$$\Delta A_1 = F\left(\frac{n}{k} - 1\right)^2 \frac{k-2}{60 + 4k}\,. \tag{54}$$

The "Fitting Parameter F" characterizes the type of a grading simplified for economical reasons. Some further details can be seen in [41].

Still more sophisticated adaption can be obtained by the following loss-*dependent* formula (see [40])

$$\Delta A_2 = \Delta A_1 \frac{1-B}{1 + k\, B^2}\,. \tag{55}$$

Eq. (55) is applied for the new dimensioning outlines of the Federal German Post Office.

4.5.3. The BQ-formula for finite number of sources [42], [41]

Because of the good results in using the MPJ-formula for PCT 1, an analogous type has been derived for PCT 2, i.e. for finite number q of sources. In a paper of ERICSSON (1955) [43] the analogue to the PJ-formula can already be found here applied to link systems using the actual call rate α per idle source to a full available route and combined with the expectation values $c(x)$ according to eq. (36).

Applying the MPJ-idea and using — instead of α — that "generating" call rate α_0 from which follows a *prescribed* carried traffic Y with ERLANG-BERNOULLI's distribution, A. BÄCHLE and U. HERZOG found the time congestion

$$E_k(\alpha_0, n, q) = \frac{E_n(\alpha, q)}{E_{n-k}(\alpha_0, q-k)} \tag{56}$$

and the call congestion

$$B_k(\alpha_0, n, q) = \frac{B_n(\alpha_0, q)}{B_{n-k}(\alpha_0, q-k)}\,. \tag{57}$$

With

$$E_n(\alpha_0, q) = \binom{q}{n} \alpha_0^n \Big/ \sum_{i=0}^{n} \binom{q}{i} \alpha_0^i \tag{58}$$

and

$$B_n(\alpha_0, q) = E_n(\alpha_0, q)\, \frac{q-n}{q-Y} \tag{59}$$

being ERLANG-BERNOULLI's distribution (mean holding time h = unity).

As in the case of the MPJ-formula, eq. (57) holds very true for good progressive gradings with skipping and may be adapted to simplified gradings analogously.

The name "BQ-formula" has been chosen from "BERNOULLI-distribution Quotient".

4.5.4. Method No. 2 of D. BOTSCH for mixed external and internal traffic [28]

The method according to Section 4.5.3 has been applied by D. BOTSCH (cf. Section 4.4.8) also to both way circuits having internal and external traffic.

His corresponding "Solution No. 2" uses the same expressions for time congestion E_k (eq. (46)) and call congestion B_k (eq. (47)) as in "Solution No. 1", but different probabilities of state $p(x)$ are chosen:

The probabilities of state $p(x)$ for "Solution No. 2" must be calculated with "generating" offered traffics $A_{0\,\mathrm{ext}}$ and $A_{0\,\mathrm{int}}$ (or call rates $\alpha_{0\,\mathrm{ext}}$ and $\alpha_{0\,\mathrm{int}}$, respectively), which lead to the prescribed carried loads Y_{ext} and Y_{int} in case of full accessibility. See also [41] and [28].

4.6. Alternate routing in case of groups with limited access

As published at the 4th ITC 1964 in London [44] and in [38], [45]—[49], the Equivalent Random Method (ERT-Method) of R. I. WILKINSON [14], [50] or G. BRETSCHNEIDER, respectively [15] could be extended to groups with limited access.

In this so-called RDA-method [44] the variance coefficient $D = V - R$ of overflow traffic behind a grading with n trunks, accessibility k, and call congestion B_k is

$$D = p\,R^2\,k/n \qquad (60)$$

with $R = AB_k$ being the "rest" of the offered traffic, i.e. the overflow traffic. The parameter $p = f(k, B_k)$ can be drawn from diagrams. Tables are available for $(R, D) = f(A, k, n)$.

The dimensioning of secondary groups, to which overflow traffic (R, D) is offered, makes use of the same idea as the ERT-method. An equivalent primary grading (EPG) has to be chosen such, that it generates the actual overflow traffic (R, D). Its ratio n^*/k^* has to be determined in a way that this EPG and the following actual secondary grading together form *one* total inhomogeneous grading, appropriate for sequential hunting. Further details about this method can be found in [51], [52]. The ERT- and RDA-method are also part of the new outlines of the Federal German Post Office.

It should be denoted that BRIDGFORD's method developed for two-stage link systems has also been applied for the special case of one-stage gradings [26].

The problem of the exact calculation of overflow systems with full or limited availability in primary and/or secondary groups has been investigated in detail by R. SCHEHRER [21], [22].

5. Gradings in Delay Systems

5.1. Interpolation methods of E. GAMBE [53]

The first step towards gradings in delay systems was done by E. GAMBE. For gradings having the availability k and the number of trunks n E. GAMBE investigated interpolation methods to find simple approximations for the delay probability W and for the mean waiting time τ_w by considering full available groups with k or n trunks, respectively.

5.2. Delay systems with ideal and nonideal gradings

5.2.1. The Interconnection Delay Formula (IDF) [54]—[56]

For ideal Erlang gradings with waiting M. THIERER developed explicit formulas for $p(x)$, W, and τ_w by application of the statistical equilibrium under a special equilibrium condition:

$$p(x) = \frac{\dfrac{A^x}{\prod\limits_{z=1}^{x}[z - c(z)\,A]}\prod\limits_{z=0}^{x-1}u(z)}{1 + \sum\limits_{j=1}^{n}\dfrac{A^j}{\prod\limits_{z=1}^{j}[z - c(z)\,A]}\prod\limits_{z=0}^{j-1}u(z)}, \quad x = 0, 1, \ldots, n, \qquad (61)$$

$$W = \sum_{x=k}^{n} p(x)\,c(x), \qquad (62)$$

$$\tau_w = \frac{\sum\limits_{x=k}^{n}\left[p(x)\sum\limits_{z=k}^{x}\dfrac{c(z)\,A}{z - c(z)\,A}\right]}{A\,W}. \qquad (63)$$

The results of artificial traffic tests with ideal gradings comply as exactly as possible with the theory. For familiar nonideal gradings the method yields good approximate values, too. Tables for $W(A)$ and $\tau_w(A)$ have been calculated according to the IDF for $k = 5$ up to 40 and $n = k$ up to 100 [56].

The solution has also been extended to the case where the holding times are constant [57].

5.2.2. The Grading Delay Formula (GDF) [56]

For nonideal gradings M. THIERER has improved the IDF by substitution of the distribution function $p(x)$ according to eq. (61) by the corresponding one of a fully available trunk group with the same number of trunks n and the same traffic carried. Traffic tests have shown that the GDF yields good approximate results.

5.3. Combined delay and loss systems with gradings [58], [59]

Allowing only a finite number of waiting places in front of each grading group P. KÜHN investigated exact calculation methods for the probabilities of state and the distribution function of waiting times. For symmetrically structured systems a recursion algorithm for the probabilities of state was derived on the basis of equilibrium equations and a special symmetry condition [59]. Comparing with traffic tests the solution yields good approximate results for ideal and nonideal gradings.

6. Conclusion

The author tried to give an abridged but systematic survey of the approximate methods for the calculation of one stage gradings, which have been developed since 1920. Because of the huge number of publications in this field, it was impossible to mention all interesting studies.

There is a large number of problems not yet solved, as for example:

a) Improved approximate methods for loss calculation, if unbalanced traffic is offered.
b) Improved methods for loss calculation in case of smooth traffic.
c) Problems of overflow traffic and alternate routing with finite number of traffic sources.

Therefore, the treatment of gradings will be an interesting field for traffic theorists even in future.

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