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ON THE EXACT CALCULATION OF LINK SYSTEMS
WITH INTERNAL OVERFLOW

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1. INTRODUCTION

In public telephone systems the long distance networks operate normally with alternate routing. In these alternate routing systems, long distance calls can be switched via several connecting paths. By means of such possibilities to connect a call via different routes the economy, flexibility and reliability of such telephone networks is superior to systems without alternate routing.

Building up a new connection it will be tried to switch this via a trunk of the shortest, most economical route (first direct route). If this connection cannot be switched via this route, e.g. all trunks of this route are busy, the call "overflows" to another route (second direct route) which lead to the destination office via a tandem office. If at least no connection via a trunk of the final route is possible, then the call is rejected, i.e. it becomes lost.

By reason of economy direct routes normally are used as high usage trunk groups, i.e. the probability of overflow is high. The final routes, on the other hand, must operate with low losses [1], [2].

Figure 1 shows a very plain overflow system, consisting of one direct route and the final route.

To switch the traffic flow in the exchanges, there are used "group selectors". Their task is to connect the offered calls to an idle outgoing trunk of an appropriate group.

The principle of such a group selector will be explained by means of the above example. Figure 2 shows such a group selector for the exchange A of Fig. 1.

At first the control of this group selector in the exchange A tries to connect a call with an outgoing trunk of the direct route AB. If this connection is impossible, e.g. all trunks of the direct route AB are busy, then the control will reverse the call to the final route ACB. The call overflows from the high usage trunk group to the final trunk group. If the call will be refused at the final route ACB, too, then it

581

becomes lost in this group selector. Such group selectors can be built up as one- or multistage connecting arrays. By reason of economy the significance of multistage connecting arrays with conjugated selection (link systems) increases rapidly.
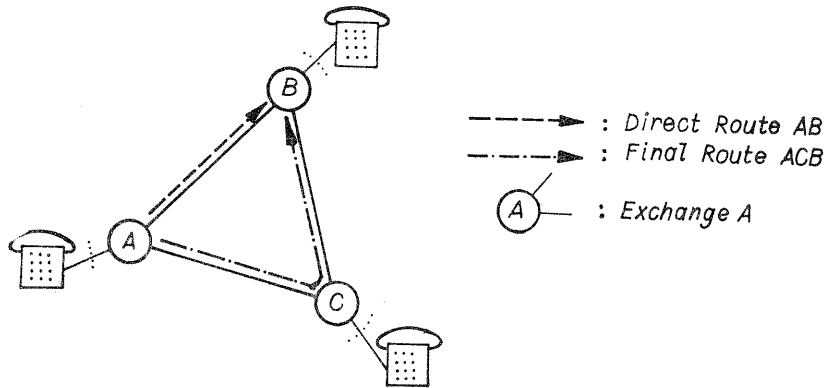

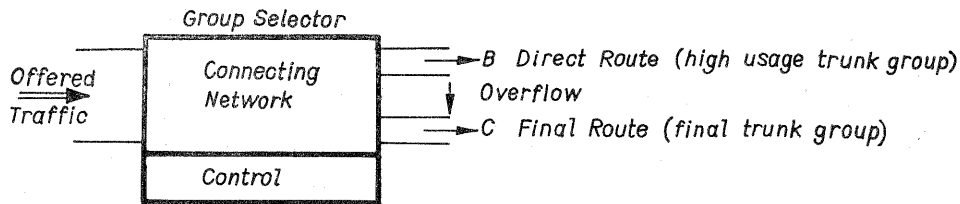Fig. 1. Example for a telephone network with alternate routing.

Fig. 2. Group selector with overflow possibility in the exchange A.

Up to now, there exists a large number of methods for approximative or exact calculation of overflow systems, but they assume one-stage connecting arrays with full or limited availability (see e.g. [3]). For the calculation of link systems exist a large number of algorithms, too [4], [5]. But for link systems with internal overflow only some approximative methods are known [6]–[9].

This paper deals with the exact calculation of link systems with internal overflow. By reason of clearness only single overflow (direct route → final route), see e.g. Fig. 1, and only two-stage link systems will be considered.

In principle systems with more than one overflow possibility (first direct route → → second direct route → ... → final route) as well as systems with more than two stages can be calculated with the same method.

The arriving time intervals and the holding times are assumed to be time independent and negative exponentially distributed. The system of linear equations for the state pattern probabilities (stationary probabilities) is derived. Already for small

two-stage link systems the number of different state patterns increases rapidly. Therefore this system of equations can be set up and solved only by means of a digital computer. Consequently, one of the main aspects in this paper is the determination of the coefficients, terms, etc. in a general programable mode.

The characteristic traffic values, such as e.g. probability of loss, carried traffic or overflow traffic are calculated from the state pattern probabilities. Example is given to demonstrate the influence of internal overflow to the characteristic traffic values.

2. STRUCTURE OF THE LINK SYSTEM AND THE OPERATION MODE

2.1. Description of the Structure

Two-stage link systems with the following structure are considered (Fig. 3):

The system is denoted by a set of parameters

i_v, k_v number of inlets resp. outlets per multiple of stage v,

R number of outgoing trunk groups,

g_v number of multiples of stage v,

$_rk_2$ number of outlets per multiple of stage 2 to group r, $r \in [1, R]$,

$_rn_2$ number of outgoing trunks to group r,

$g_1 \cdot k_1 \geqq g_2 \cdot i_2$ where $>$ grading between the outlets of stage 1,
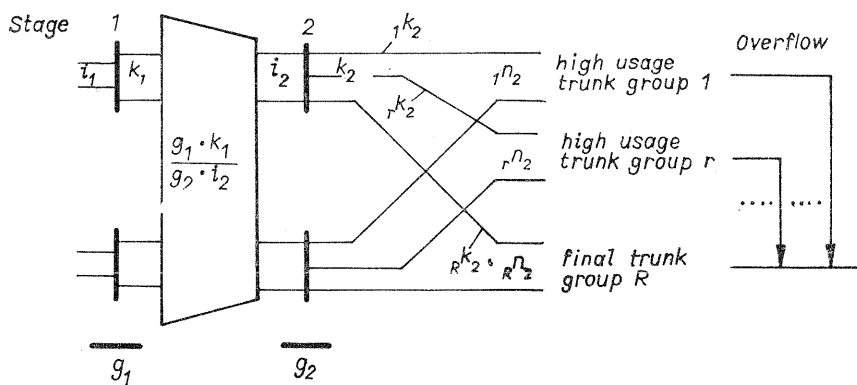$\qquad\qquad\qquad$ $=$ without grading.

Fig. 3. Two-stage link system with internal overflow.

583

We introduce the following abbreviated notations:

outlet (w, v, v): outlet w in multiple v of stage v

multiple (v, v): multiple v of stage v

To describe the structure of the link system, we define a "grading matrix Θ" with the dimension $(g_1 \times k_1)$. This matrix denotes the connecting network between the outlets of stage 1 and the inlets of stage 2. The element ϑ_{vw} characterizes the outlet $(w, v, 1)$. Its numerical value indicates the number of this inlet of stage 2 which is wired with the outlet $(w, v, 1)$, i.e. $\vartheta_{vw} \in [1, g_2 . i_2]$.

From this grading matrix Θ, we can derive a "multiple matrix Φ" with the same dimension. The numerical value of φ_{vw} indicates the number of the multiple of stage 2 wired with the outlet $(w, v, 1)$, i.e. $\varphi_{vw} \in [1, g_2]$. If the inlets of stage 2 are numbered in sequential order, then φ_{vw} is given by

(1)
$$\varphi_{vw} = \left(\frac{\vartheta_{vw} - 1}{i_2} + 1\right)_{\text{rounded}}.$$

2.2. Operation Mode

The operation mode is as follows:

A call arriving in multiple $(v, 1)$ should be connected to an outlet of the outgoing trunk group r. We must consider two cases:

$r \in [1, R)$: If there is no trunk of this group r (high usage trunk group) idle or if there is no idle path through the system to such a free trunk (internal blocking), then this call overflows to the final trunk group R.

$r = R$: If the call offered (directly offered or overflowing from a high usage trunk group) to group R cannot be connected to an outlet of this group, then it becomes lost.

The hunting mode in the multiples of stage 1 is assumed to be

random hunting: all idle outlets in the multiple will be occupied with the same probability, or

sequential hunting: starting at a fixed home position the outlets in the multiple will be hunted in sequential order.

The outlets in the multiples of stage 2 will be hunted sequentially.

2.3. Occupation State

2.3.1. State Pattern $\{\tilde{x}\}$

To describe the occupation state of the link system we introduce the state pattern $\{\tilde{x}\}$. One state pattern $\{\tilde{x}\}$ must characterize one occupation state uniquely. Therefore it is necessary to denote for each connection through the link system

— the occupied outlet $(w, v, 1)$,

— the destination route r,

— the outgoing trunk group r or R.

As shown in $[10]$ we introduce a "state matrix S" with the dimension $(g_1 \times k_1)$. It is

(2) $s_{vw} = r^*$ if the outlet $(w, v, 1)$ is busy to destination route r,

            $= 0$ if this outlet is not busy,

where

(3) $r = r^*_{\mathrm{mod}R}$ with $r^* \in [1, 2R)$ and $r \in [1, R]$.

By means of this definition s_{vw} denotes the connection through the link system uniquely:

$s_{vw} = r^* < R$ the outlet $(w, v, 1)$ is connected with an outgoing trunk of a high
            usage trunk group,

        $\geq R$ the outlet $(w, v, 1)$ is connected with an outgoing trunk of the final
            trunk group.

2.3.2. Neighbouring State Patterns $\{\tilde{x} + 1\}$ and $\{\tilde{x} - 1\}$

"Neighbouring patterns" are all these patterns where the state matrix differs only in one element to the matrix S of the state pattern $\{\tilde{x}\}$ without regard to the realization of such a pattern.

"Neighbouring state patterns" $\{\tilde{x} + 1\}$ resp. $\{\tilde{x} - 1\}$ differ from the state pattern $\{\tilde{x}\}$ in one connection. The state pattern $\{\tilde{x}\}$ has x connections, then the

"upper neighbouring state pattern" (UNS-pattern) $\{\tilde{x} + 1\}$ has $(x + 1)$ and the

"lower neighbouring state pattern" (LNS-pattern) $\{\tilde{x} - 1\}$ has $(x - 1)$ connections.

The neighbouring state patterns as well as the neighbouring patterns can be characterized by means of the "Kronecker Symbol":

(4)
$$\delta_{xy} = 1 \quad \text{if} \quad x = y \,,$$
$$= 0 \quad \text{if} \quad x \neq y \,,$$

the UNS-pattern $\{\tilde{x} + 1\}$ with $\|s_{ij} + r^*\delta_{iv}\delta_{jw}\|$,

LNS-pattern $\{\tilde{x} - 1\}$ with $\|s_{ij} - r^*\delta_{iv}\delta_{jw}\|$,

where only the occupation of the outlet $(w, v, 1)$ differs between $\{\tilde{x}\}$ and $\{\tilde{x} + 1\}$ resp. $\{\tilde{x} - 1\}$.

3. TRAFFIC

Numerous measurements in different countries have shown that during the busy hours telephone traffic is a stationary process, where the arriving time intervals and the holding times are negative exponentially distributed.

Therefore two types of offered traffic can be defined:

PCT 1: Pure chance traffic of type 1.

An infinite number of independent sources produces the traffic with the constant call rate $_r\lambda_v$.

PCT 2: Pure chance traffic of type 2.

A finite number of independent sources produces the traffic; the total call rate $_r\lambda_v$ depends on the number of busy sources. Each idle source has a call rate $_r\alpha_v$.

The call rate $_r\lambda_v$ resp. $_r\alpha_v$ and also the termination rate $_r\mu_v$ can be different for the multiples $(v, 1)$, $v \in [1, g_1]$ and for the destination routes r, $r \in [1, R]$.

Let be

(5)
$$\left. \begin{array}{l} s^*_{vw} = 1 \\ = 0 \end{array} \right\} \quad \text{if} \quad s_{vw} \begin{cases} > 0 \,, \\ = 0 \,, \end{cases}$$

s^*_{vw} denotes if the outlet $(w, v, 1)$ is busy or not,

(6)
$$s^*_{v.} = \sum_{w=1}^{k_1} s^*_{vw}$$

number of busy outlets in multiple $(v, 1)$,

$_r\alpha_v$ the call rate to destination route r of each idle source connected with the multiple $(v, 1)$,

(7a)
$$_r\lambda_v = {_r\alpha_v}(i_1 - s^*_{v.}) \quad \text{if} \quad \text{PCT 2} \,;$$

the call rate to destination route r of all idle sources connected with the multiple $(v, 1)$,

(7b) $$_r\lambda_v = \text{const} \quad \text{if} \quad \text{PCT 1},$$

(8) $$_r\mu_v = 1/_rh_v$$

the termination rate of each call established from multiple $(v, 1)$ to destination route r $\left(_rh_v = \text{mean holding time}\right)$

4. SYSTEM OF EQUATIONS
FOR THE STATE PATTERN PROBABILITIES

4.1. Derivation of the System of Equations in a General Form

Assuming PCT 1 resp. PCT 2 it can be proved that the stationary (and therefore time independent) state pattern probability $p(\tilde{x})$ exists [11].

Therefore the "statistical equilibrium" holds and $p(\tilde{x})$ can be calculated from a linear system of equations, the Kolmogorov-forward-equation

(9a) $$q(\tilde{x}) \cdot p(\tilde{x}) = \sum_{\tilde{y} \neq \tilde{x}} q(\tilde{y}, \tilde{x}) \cdot p(\tilde{y})$$

where

$$q(\tilde{x}) = \sum_{\tilde{y} \neq \tilde{x}} q(\tilde{x}, \tilde{y}) .$$

$q(\tilde{y}, \tilde{x})$ is the transition coefficient from the state pattern $\{\tilde{y}\}$ to the state pattern $\{\tilde{x}\}$.

This homogeneous system of linear equations is normalized by the condition

(10) $$\sum_{\tilde{x}} p(\tilde{x}) = 1 .$$

Here, eq. (9a) can be written in a more detailed form as

(9b) $$p(\tilde{x}) \left[\lambda(\tilde{x}) + \mu(\tilde{x})\right] = \sum_{\tilde{x}+1} p(\tilde{x} + 1) \cdot \mu(\tilde{x} + 1) + \sum_{\tilde{x}-1} p(\tilde{x} - 1) \cdot \lambda(\tilde{x} - 1)$$

where

$\lambda(\tilde{x})$ the total call rate in the state pattern $\{\tilde{x}\}$,

$\mu(\tilde{x})$ the total termination rate in the state pattern $\{\tilde{x}\}$.

The four terms in eq. (9b) correspond to the four transitions from and into the state pattern $\{\tilde{x}\}$.

587

The following sections 4.2, ..., 4.5 deal with the description of these transitions in a detailed form. The notation is given in such a general form, that for any state pattern $\{\tilde{x}\}$ the neighbouring state patterns and the transition coefficients can be determined. This is necessary to solve the problem on a digital computer.

4.2. Transition $\{\tilde{x} + 1\} \rightarrow \{\tilde{x}\}$

As shown in section 2.3.2 the UNS-pattern $\{\tilde{x} + 1\}$ can be described by $\|s_{ij} + r^*\delta_{iv}\delta_{jw}\|$. All UNS-patterns $\{\tilde{x} + 1\}$ and their individual coefficient for the transition into the state pattern $\{\tilde{x}\}$ are determined in two steps:

1. Formal determination of the upper neighbouring patterns without regard to their realization.

2. Elimination of all those patterns which are not UNS-patterns.

4.2.1. Determination of the Upper Neighbouring Patterns

With the description given in section 2.3.2 we find all those patterns by summation

(11)
$$\sum_{r^*=1}^{2R-1} \sum_{v=1}^{g_1} \sum_{w=1}^{k_1} \{\|s_{ij} + r^*\delta_{iv}\delta_{jw}\|\} .$$

4.2.2. Determination of the UNS-Patterns $\{x + 1\}$

All patterns of expression (11) which satisfy the following conditions are UNS-patterns; the remaining part of the patterns is excluded by these factors.

In the state pattern $\{\tilde{x}\}$

(a) at least one source of the considered multiple $(v, 1)$ is idle:

This can be determined by means of the occupied outlets in this multiple $(v, 1)$

(12)
$$\begin{aligned} \psi_v &= 1 \\ &= 0 \end{aligned} \right\} \quad \text{if} \quad s_{v.}^* \begin{cases} < i_1 , \\ = i_1 , \end{cases}$$

where $s_{v.}^*$ acc. to (6), i.e. $\psi_v = 1$ if at least one source connected with multiple $(v, 1)$ is idle.

(b) the considered outlet $(w, v, 1)$ is not busy.

This can be taken into account by the factor $(1 - s_{vw}^*)$.

(c) the considered outlet $(w, v, 1)$ is not blocked via the grading between the outlets of stage 1.

For this purpose the occupation state of all outlets of stage 1 which are interconnected with the outlet $(w, v, 1)$ have to be examined.

$$(13) \qquad \varrho_{vw} = \sum_{\substack{y=1 \\ y \neq v}}^{g_1} \sum_{z=1}^{k_1} s_{yz}^* \cdot \delta_{\vartheta_{vw}\vartheta_{yz}} \, .$$

If $(1 - \varrho_{vw}) = 1$, then the outlet $(w, v, 1)$ is not blocked via the grading.

(d) the considered outlet $(w, v, 1)$ is not backward blocked to group r.

If all outlets in multiple $(\varphi_{vw}, 2)$ to group r are busy, then the outlet $(w, v, 1)$ is backward blocked to group r, i.e. $_{r*}\tau_{vw} = 1$.

$$(14) \qquad \begin{matrix} _{r*}\tau_{vw} = 1 \\ = 0 \end{matrix} \Bigg\} \quad \text{if} \quad (i) \ r^* < R : \qquad \sum_{y=1}^{g_1} \sum_{z=1}^{k_1} \delta_{s_{yz},r} \cdot \delta_{\varphi_{vw},\varphi_{yz}} \begin{cases} = {}_rk_2 , \\ < {}_rk_2 , \end{cases}$$

$$(ii) \ r^* \geqq R : \sum_{r^{**}=R}^{2R-1} \sum_{y=1}^{g_1} \sum_{z=1}^{k_1} \delta_{s_{yz},r^{**}} \cdot \delta_{\varphi_{vw},\varphi_{yz}} \begin{cases} = {}_Rk_2 , \\ < {}_Rk_2 ; \end{cases}$$

where φ_{vw} acc. to (1), r acc. to (3), $r^{**} \in [R, 2R)$. If $(1 - {}_{r*}\tau_{vw}) = 1$ then the outlet $(w, v, 1)$ is not backward blocked to group r.

4.2.3. Transition Rate

With the conditions (a), ..., (d) from section 4.2.2 the transition rate from all UNS-patterns $\{\tilde{x} + 1\}$ into the state pattern $\{\tilde{x}\}$ can be determined as

$$(15) \qquad \sum_{r^*=1}^{2R-1} \sum_{v=1}^{g_1} \sum_{w=1}^{k_1} p(\|s_{ij} + r^*\delta_{iv}\delta_{jw}\|) \, \psi_v(1 - s_{vw}^*) \, (1 - \varrho_{vw}) \, (1 - {}_{r*}\tau_{vw}) \, {}_r\mu_v$$

where r acc. to (3), ψ_v acc. to (12), ϱ_{vw} acc. to (13), $_{r*}\tau_{vw}$ acc. to (14), $_r\mu_v$ acc. to (8).

4.3. Transition $\{\tilde{x}\} \rightarrow \{\tilde{x} + 1\}$

The outlet $(w, v, 1)$ is backward blocked to destination route r, i.e. to the high usage trunk group r and to the final trunk group R, if $_r\tau_{vw} = 1$.

$$(16) \qquad \begin{matrix} _r\tau_{vw} = 1 \\ = 0 \end{matrix} \Bigg\} \quad \text{if} \quad (i) \ r < R: \ \sum_{r^{**}=1}^{2R-1} \sum_{y=1}^{g_1} \sum_{z=1}^{k_1} (\delta_{s_{yz},r} + \delta_{s_{yz},r^{**}}) \cdot \delta_{\varphi_{vw},\varphi_{yz}} \begin{cases} = {}_rk_2 + {}_Rk_2 , \\ < {}_rk_2 + {}_Rk_2 , \end{cases}$$

$$(ii) \ r = R: \ \sum_{r^{**}=R}^{2R-1} \sum_{y=1}^{g_1} \sum_{z=1}^{k_1} \delta_{s_{yz},r^{**}} \cdot \delta_{\varphi_{vw},\varphi_{yz}} \qquad \begin{cases} = {}_Rk_2 , \\ < {}_Rk_2 , \end{cases}$$

where $r^{**} \in [R, 2R)$, r acc. to (3).

The outlet $(w, v, 1)$ is blocked to destination route r if it is

- not busy, i.e. $\left(1 - s_{vw}^*\right) = 1$,

- either blocked via the grading, i.e. $\varrho_{vw} = 1$,

- or not blocked via the grading, i.e. $\left(1 - \varrho_{vw}\right) = 1$, but backward blocked to group r and R, i.e. $_r\tau_{vw} = 1$

$$(17) \qquad _r\omega_{vw} = \left(1 - s_{vw}^*\right)\left[\varrho_{vw} + \left(1 - \varrho_{vw}\right)_r\tau_{vw}\right],$$

i.e. $_r\omega_{vw} = 1$, if the outlet $(w, v, 1)$ is blocked to destination route r.

The number of blocked outlets to destination route r in multipe $(v, 1)$ is given by

$$(18) \qquad _r\omega_{v.} = \sum_{w=1}^{k_1} {}_r\omega_{vw}.$$

The transition into an UNS-pattern $\{\tilde{x} + 1\}$ from the state pattern $\{\tilde{x}\}$ is possible if at least

(a) one source of the considered multiple $(v, 1)$ is idle.

This condition is contained in $_r\lambda_v$ acc. to (7).

(b) one outlet of the considered multiple $(v, 1)$ is idle to destination route r.

If the number of outlets in multiple $(v, 1)$ which are busy or blocked to destination route r is less than k_1, then such an idle outlet, i.e. $\left(1 - {}_r\xi_v\right) = 1$ does exist.

$$(19) \qquad \left.\begin{array}{l} _r\xi_v = 1 \\ = 0 \end{array}\right\} \quad \text{if} \quad s_{v.}^* + {}_r\omega_{v.} \left\{\begin{array}{l} = k_1, \\ < k_1, \end{array}\right.$$

where $s_{v.}^*$ acc. to (6), $_r\omega_{v.}$ acc. to (18).

With these conditions (a) and (b) the transition rate from the state pattern $\{\tilde{x}\}$ into all UNS-patterns $\{\tilde{x} + 1\}$ can be determined as

$$(20) \qquad p\left(\|s_{ij}\|\right) \sum_{r=1}^{R} \sum_{v=1}^{g_1} \left(1 - {}_r\xi_v\right) \cdot {}_r\lambda_v,$$

where $_r\lambda_v$ acc. to (7), $_r\xi_v$ acc. to (19).

4.4. Transition $\{\tilde{x}\} \to \{\tilde{x} - 1\}$

A transition from the state pattern $\{\tilde{x}\}$ into a LNS-pattern $\{\tilde{x} - 1\}$ is possible if the outlet $(w, v, 1)$ is busy. In this case the connection via this outlet can terminate.

Therefore the transition rate from the state pattern $\{\tilde{x}\}$ into all LNS-patterns is given by

$$(21) \qquad p(\|s_{ij}\|) \sum_{r^*=1}^{2R-1} \sum_{v=1}^{g_1} \sum_{w=1}^{k_1} \delta_{s_{vw},r^*} \cdot {}_r\mu_v$$

where ${}_r\mu_v$ acc. to (8).

4.5. Transition $\{\tilde{x}-1\} \rightarrow \{\tilde{x}\}$

The procedure to determine the transition rate from all LNS-patterns $\{\tilde{x}-1\}$ into the state pattern $\{\tilde{x}\}$ is similar to section 4.2:

1. Formal determination of the lower neighbouring patterns without regard to their realization.

2. Elimination of all those patterns which are not LNS-patterns.

4.5.1. Determination of the Lower Neighbouring Patterns

Firstly we can find the patterns by the same summations as in expression (11). However, we can exclude a large number of patterns by replacing k_1 with ${}_{r^*}\varepsilon_v$. The value of ${}_{r^*}\varepsilon_v$ depends on the hunting mode. Therefore we can distinguish two cases:

A. Sequential hunting

Let be $(w'+1)$ the first — sequentially hunted — idle outlet in multiple $(v,1)$ to group r in the state pattern $\{\tilde{x}\}$. If a source connected with this multiple becomes busy in the lower neighbouring pattern, than this source can be connected only to an outlet which is sequential less then or equal to $(w'+1)$. If the outlet $(w'+1)$ will be occupied by this call, then another state pattern originates as $\{\tilde{x}\}$. Therefore the summation can be limited up to ${}_{r^*}\varepsilon_v = w'$.

To determine ${}_{r^*}\varepsilon_v$ it is necessary to define a coefficient ${}_{r^*}\omega_{vz}$. It is ${}_{r^*}\omega_{vz} = 1$, if the outlet $(z,v,1)$ is blocked to group r (high usage trunk group).

Similar to (17) we obtain

$$(22) \qquad {}_{r^*}\omega_{vz} = \left(1 - s_{vz}^*\right)\left[\varrho_{vz} + \left(1 - \varrho_{vz}\right) \cdot {}_{r^*}\tau_{vz}\right],$$

where ϱ_{vz} acc. to (13), ${}_{r^*}\tau_{vz}$ acc. to (14).

The number of blocked outlets to group r in multiple $(v,1)$ is given by

$$(23) \qquad {}_{r^*}\omega_{v.} = \sum_{z=1}^{k_1} {}_{r^*}\omega_{vz}.$$

We obtain

(i) $r^* \leqq R$

(24a)
$$_{r*}\varepsilon_v = \inf\left(\varepsilon^* \bigg| \sum_{z=1}^{\varepsilon^*+1} {}_{r*}u_{vz} < \varepsilon^* + 1, \ \varepsilon^* \in [0, k_1]\right),$$

where

(25)
$$\begin{aligned} {}_{r*}u_{vz} &= s_{vz}^* + {}_{r*}\omega_{vz} \quad \text{if} \quad z \in [1, k_1], \\ &= 0 \quad\quad\quad\quad \text{if} \quad z = k_1 + 1, \end{aligned}$$

and

(26)
$$_{r}\bar{\varepsilon}_v = {}_{r*}\varepsilon_v$$

with r acc. to (3).

(ii) $r^* > R$. Internal overflow to the final trunk group R is impossible if there is an idle outlet to group r in multiple $(v, 1)$, i.e. if $_r\bar{\varepsilon}_v < k_1$.

Hence:

(24b)
$$\left.\begin{aligned} {}_{r*}\varepsilon_v &= \inf\left(\varepsilon^* \bigg| \sum_{z=1}^{\varepsilon^*+1} {}_{r*}u_{vz} < \varepsilon^* + 1, \ \varepsilon^* \in [0, k_1]\right) \\ &\quad \text{where } {}_{r*}u_{vz} \text{ acc. to (25)} \\ &= 0 \\ &\quad \text{where } {}_r\bar{\varepsilon}_v \text{ acc. to (26).} \end{aligned}\right\} \text{if } {}_r\bar{\varepsilon}_v \left\{\begin{aligned} &= k_1, \\ &< k_1, \end{aligned}\right.$$

B. Random hunting

In this case, the occupation of each idle outlet in the multiple is possible. Therefore we obtain

(i) $r^* \leqq R$

(27a)
$$_{r*}\varepsilon_v = k_1.$$

If there is no idle outlet to group r in the multiple $(v, 1)$, then internal overflow to the final trunk group R occurs.

Let

(28)
$$\left.\begin{aligned} {}_r\xi_v^* &= 1 \\ &= 0 \end{aligned}\right\} \text{if} \quad s_{v.}^* + {}_{r*}\omega_v. \left\{\begin{aligned} &= k_1, \\ &< k_1, \end{aligned}\right.$$

where $s_{v.}^*$ acc. to (6), $_{r*}\omega_v.$ acc. to (23), i.e. $_r\xi_v^* = 1$, if all outlets in multiple $(v, 1)$ are busy or blocked to group r (high usage trunk group).

(ii) $r^* > R$

(27b)
$$_{r^*}\varepsilon_v = k_1 \Big\} \quad \text{if} \quad _r\zeta_v^* \begin{cases} = 1, \\ = 0. \end{cases}$$
$$\qquad\qquad = 0 \Big\}$$

With this upper boundary value of the summation $_{r^*}\varepsilon_v$ we obtain the lower neighbouring patterns

(29)
$$\sum_{r^*=1}^{2R-1} \sum_{v=1}^{g_1} \sum_{w=1}^{r^*\varepsilon_v} \left\{ \left\| s_{ij} - r^* \delta_{iv} \delta_{jw} \right\| \right\} .$$

4.5.2. Determination of the LNS-Patterns $\{\tilde{x} - 1\}$

All patterns of expression (29) satisfying the following conditions are LNS-patterns. The corresponding factors must be defined in such a way, that the remaining part of the patterns is excluded.

(a) The considered outlet $(w, v, 1)$ is occupied to group r resp. R in the state pattern $\{\tilde{x}\}$.

This condition can be realized with a factor δ_{s_{vw}, r^*}. The numerical value of δ_{s_{vw}, r^*} is unequal to 0 only if the outlet $(w, v, 1)$ is connected with an outgoing trunk to group r resp. R.

(b) In the lower neighbouring pattern it must be possible that a call offered to group r in multiple $(v, 1)$ can be connected through the link system via the outlet $(w, v, 1)$.

We introduce a factor $_{r^*}\eta_{vw}$. This is the probability that such a call is connected via this outlet $(w, v, 1)$; it depends on the hunting mode. Therefore we must distinguish two cases:

A. Sequential hunting

Here, there are two different ranges, too:

(i) $r^* \leq R$. In this case it must be checked that the considered outlet $(w, v, 1)$ is the first sequentially hunted idle outlet in multiple $(v, 1)$ to group r in the neighbouring lower pattern.

For this purpose we must consider all outlets $(\zeta < w)$ and determine if there is an outlet ζ blocked to group r in the state pattern $\{\tilde{x}\}$ but idle to this group in the LNS-pattern $\{\tilde{x} - 1\}$:

The outlet ζ is blocked in such a way if it is

— wired with the same multiple of stage 2 as the outlet w, i.e. $\varphi_{v\zeta} = \varphi_{vw}$
— not busy, i.e. $\left(1 - s_{v\zeta}^*\right) = 1$
— not blocked via the grading, i.e. $\left(1 - \varrho_{v\zeta}\right) = 1$
— backward blocked to group r

The last condition is already included in $_{r^*}\varepsilon_v$.

Hence:

$$(30) \qquad \beta_{vw,\zeta} = \left(1 - s_{v\zeta}^*\right)\left(1 - \varrho_{v\zeta}\right)\delta_{\varphi_{v\zeta}\varphi_{vw}} \, ,$$

i.e. $\beta_{vw,\zeta} = 1$, if the outlet ζ is blocked by the outlet w in the state pattern $\{\tilde{x}\}$.

If there is such an outlet in the multiple $(v, 1)$, then $\beta_{vw} = 1$.

$$(31) \qquad \left. \begin{matrix} \beta_{vw} = 1 \\ = 0 \end{matrix} \right\} \quad \text{if} \quad \sum_{\zeta=1}^{w-1} \beta_{vw,\zeta} \left\{ \begin{matrix} > 0, \\ = 0, \end{matrix} \right.$$

We obtain

$$(32a) \qquad _{r^*}\eta_{vw} = \left(1 - \beta_{vw}\right).$$

Furthermore we must determine the number of busy outgoing trunks to group r (high usage trunk group) in the multiple $(\varphi_{vw}, 2)$. Only in the case that all these outgoing trunks are busy a new occupation of an outgoing trunk to group R (final trunk group) is possible, i.e. $_r\tau_{vw}^* = 1$.

$$(33) \qquad _r\tau_{vw}^* = _{r^*}\tau_{vw} \quad \text{for} \quad r^* \in \left[1, R\right].$$

(ii) $r^* > R$. Here, in addition to the conditions of (i) it must be checked, that there is no idle outlet in multiple $(v, 1)$ to group r in the LNS-pattern.

If this condition is true, then the lower neighbouring pattern is a LNS-pattern.

Hence:

$$(34) \qquad _{r^*}\beta_{vw} = \beta_{vw} \cdot _r\tau_{vw}^* + \left(1 - _r\tau_{vw}^*\right),$$

where β_{vw} acc. to (31), $_r\tau_{vw}^*$ acc. to (33), r acc. to (3), and

$$(32b) \qquad _{r^*}\eta_{vw} = \left(1 - _{r^*}\beta_{vw}\right).$$

B. Random hunting

(i) $r^* \leqq R$. Here all outlets of the multiple $(v, 1)$ are considered. We must determine the number of outlets which are idle to group r in the LNS-pattern.

The number of outlets in multiple $(v, 1)$ which are blocked to group r in the state pattern $\{\tilde{x}\}$ but idle to this group in the LNS-pattern is given by

$$(35) \qquad _{r^*}\beta_{vw} = \sum_{\substack{\zeta=1 \\ \zeta \neq w}}^{k_1} \beta_{vw,\zeta} \cdot _{r^*}\tau_{v\zeta} \, ,$$

where $\beta_{vw,\zeta}$ acc. to (30), $_{r^*}\tau_{v\zeta}$ acc. to (14).

Then the probability that the new successful call in multiple $(v, 1)$ offered to group r in the LNS-pattern is connected to the considered outlet w is

$$(36a) \qquad _{r^*}\eta_{vw} = \left(k_1 - s_{v.}^* - _{r^*}\omega_{v.} + 1 + _{r^*}\beta_{vw}\right)^{-1} \, ,$$

where $s_{v.}^*$ acc. to (6), $_{r*}\omega_{v.}$ acc. to (23), $_{r*}\beta_{vw}$ acc. to (35), i.e. $_{r*}\eta_{vw}^{-1}$ is the number of idle outlets to group r in multiple $(v, 1)$ in the LNS-pattern.

(ii) $r^* > R$. The condition that all outgoing trunks to group r (high usage trunk group) in multiple $(\varphi_{vw}, 2)$ are busy is an additional assumption which must be regarded.

Hence:

$$(36b) \qquad _{r*}\eta_{vw} = \frac{_r\tau_{vw}^*}{k_1 - s_{v.}^* - _{r*}\omega_{v.} + 1 + _{r*}\beta_{vw}}$$

where $_r\tau_{vw}^*$ acc. to (33), $_{r*}\omega_{v.}$ acc. to (23), $s_{v.}^*$ acc. to (6), $_{r*}\beta_{vw}$ acc. to (35), i.e. $_{r*}\eta_{vw} > 0$, if all outgoing trunks to group r in multiple $(\varphi_{vw}, 2)$ are busy.

4.5.3. Transition Rate

With the conditions (a) and (b) resp. with the coefficient $_{r*}\eta_{vw}$ from section 4.5.2 the transition rate from all LNS-patterns $\{\tilde{x} - 1\}$ into the state pattern $\{\tilde{x}\}$ can be determined by

$$(37) \qquad \sum_{r^*=1}^{2R-1} \sum_{v=1}^{g_1} \sum_{w=1}^{r*\varepsilon_v} p(\|s_{ij} - r^*\delta_{iv}\delta_{jw}\|) \, \delta_{s_{vw}, r^*} \cdot {}_{r*}\eta_{vw} \cdot {}_r\lambda_v^*$$

with

$$(38a) \qquad _r\lambda_v^* = {}_r\alpha_v\big(i_1 - s_{v.}^* + 1\big) \quad \text{if} \quad \text{PCT 2},$$

$$(38b) \qquad \qquad = {}_r\lambda_v \qquad\qquad \text{if} \quad \text{PCT 1},$$

$_{r*}\eta_{vw}$ acc. to (32) if sequential hunting or acc. to (36) if random hunting.

4.6. System of Equations

According to eq. (9b) the system of equations is obtained with the transition rates (15), (16), (21) and (37)

$$(39) \qquad \sum_{r^*=1}^{2R-1} \sum_{v=1}^{g_1} \Big[\sum_{w=1}^{k_1} p(\|s_{ij} + r^*\delta_{iv}\delta_{jw}\|) \, \psi_v\big(1 - s_{vw}^*\big)\big(1 - \varrho_{vw}\big)\big(1 - {}_{r*}\tau_{vw}\big) \, {}_r\mu_v +$$

$$+ \sum_{w=1}^{r*\varepsilon_v} p(\|s_{ij} - r^*\delta_{iv}\delta_{jw}\|) \, \delta_{s_{vw}, r^*} \cdot {}_{r*}\eta_{vw} \cdot {}_r\lambda_v^* \Big] -$$

$$- p(\|s_{ij}\|) \sum_{v=1}^{g_1} \Big[\sum_{r=1}^{R} \big(1 - {}_r\xi_v\big) \, {}_r\lambda_v + \sum_{r^*=1}^{2R-1} \sum_{w=1}^{k_1} \delta_{s_{vw}, r^*} \cdot {}_r\mu_v \Big] = 0$$

normalized by the condition (10):

(40)
$$\sum_S p(S) = 1$$

where $\sum_S$ comprises all state patterns of the system.

5. CHARACTERISTIC TRAFFIC VALUES

Characteristic traffic values are e.g. the probability of loss, the probability of overflow, the carried traffic, the offered traffic and the overflow traffic. They can be calculated from the state pattern probabilities acc. to eq. (39) and (40).

5.1. Probability of Loss $_rB$

The probability of loss $_rB$ for calls offered to destination route r is given by

(41)
$$_rB = \frac{\sum_S [p(S) \sum_{v=1}^{g_1} {}_r\xi_v \cdot {}_r\lambda_v]}{\sum_S [p(S) \sum_{v=1}^{g_1} {}_r\lambda_v]},$$

where $_r\xi_v$ acc. to (19), $_r\lambda_v$ acc. to (7).

5.2. Probability of Overflow $_rB^*$

The probability of overflow $_rB^*$ from the high usage trunk group r to the final trunk group R is acc. to eq. (41), where $_r\xi_v$ is replaced by $_r\xi_v^*$ acc. to (28).

5.3. Carried Traffic

The carried traffic $_rY^*$ per high usage trunk group r is

(42)
$$_rY^* = \sum_S [p(S) \sum_{v=1}^{g_1} \sum_{w=1}^{k_1} \delta_{S_{vw},r}]$$

and the total carried traffic $_rY$ to destination route r is acc. to eq. (42), where

$$\delta_{S_{vw},r} \text{ is replaced by } \delta_{S_{vw},r} + \delta_{S_{vw},(R+r)} \,.$$

5.4. Offered Traffic $_rA$

The offered traffic $_rA$ to destination route r is

$$(43) \qquad _rA = \sum_s p(s) \sum_{v=1}^{g_1} \frac{_r\lambda_v}{_r\mu_v} .$$

5.5. Overflow Traffic $_rR$

The overflow traffic $_rR$ from the high usage trunk group r to the final trunk group R is

$$(44) \qquad _rR = \sum_s p(s) \sum_{v=1}^{g_1} \frac{_r\zeta_v^* \cdot _r\lambda_v}{_r\mu_v} .$$

5.6. Other Characteristic Traffic Values

All other characteristic traffic values, e.g. the probability of loss for the final trunk group R or the carried traffic per multiple (v, v) etc., can be calculated in a similar way

6. NUMERICAL INVESTIGATIONS

This prescribed method for the exact calculation of link systems with internal overflow had been programmed for two-stage link systems [12].
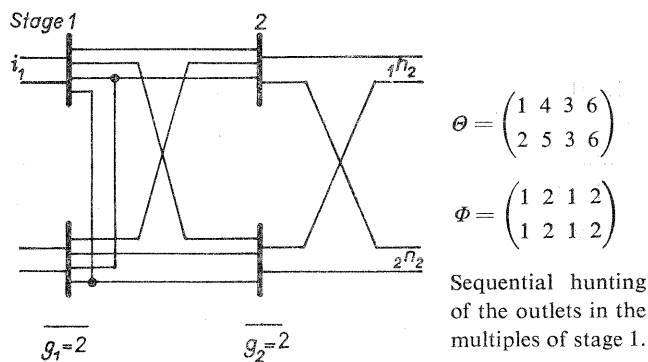

$$\Theta = \begin{pmatrix} 1 & 4 & 3 & 6 \\ 2 & 5 & 3 & 6 \end{pmatrix}$$

$$\Phi = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

Sequential hunting of the outlets in the multiples of stage 1.

Fig. 4. Two-stage link system with grading between the outlets of stage 1.

TABLE 1

Probability of loss $_rB$ to destination route r for example 1

$_rB$ \\ i_1	$_1B$	$_2B$
3	$1.7358 . 10^{-3}$	$5.3561 . 10^{-2}$
6	$4.4955 . 10^{-2}$	$2.1924 . 10^{-1}$
12	$2.2879 . 10^{-1}$	$5.0944 . 10^{-1}$

TABLE 2

Probability of loss $_rB$ to destination route r for example 2

$_rB$	$_1B$	$_2B$
2	$1.8248 . 10^{-3}$	$5.1349 . 10^{-3}$
5	$4.0445 . 10^{-2}$	$8.0078 . 10^{-2}$

Up to now some two-stage link systems have been calculated. The following two examples show as a characteristic traffic value the probability of loss as a function of structure parameters and traffic parameters.

EXAMPLE 1. Here we investigate the dependence between the probability of loss $_rB$ to destination route r and the number of sources per multiple of stage 1.

Figure 4 shows this link system and its formal description. The number of outgoing trunks to group 1 resp. 2 are $_1n_2 = {}_2n_2 = 2$. For given values of the call rates $_r\alpha_1 = 4$, $_r\alpha_2 = 2$ and of the termination rates $_r\mu_v = \mu = 2$ we obtain the following results (Tab. 1).

EXAMPLE 2. In this example we consider the link system of example 1 (Fig. 4) where the number of outgoing trunks to group 1 and 2 differs, the direct route 1 has $_1n_2 = 2$, the final route $_2n_2 = 4$ trunks. There are 6 sources per multiple of stage 1, with the same call rate $_r\alpha_v = \alpha$ and the termination rate $_r\mu_v = \mu = 30$.

We get the following results (Tab. 2).

7. CONCLUSION

By means of the described method link systems with internal overflow can be calculated exactly.

Now the problem is that the number of different state patterns and therefore the order of the system of equations for the state pattern probabilities increases rapidly if the link system grows up. Therefore only small link systems can be calculated, e.g. there are 1089 different state patterns in example 1 $(i_1 = 6)$ or 5929 different state patterns in example 2 $({}_1n_2 = 2, {}_2n_2 = 4)$.

With some assumptions of symmetry it is possible to reduce the order of the system of equations in a similar way as shown in [10] for two-stage link systems with the operation mode "preselection" and random hunting of the outlets in the multiples of stage 1. But up to now this problem is neither for preselection nor for internal overflow solved in a general form.

REFERENCES

[1] A. Lotze: Problems of traffic theory in the design of international direct distance dialling networks. Nachrichtentech. Z. (Commun. J.) *7* (1968), 2/3, 41—46.

[2] A. Lotze, R. Schehrer: The design of alternate routing systems with regard to the variance coefficient. Nachrichtentech. Z. (Commun. J.) *7* (1968), 2/3, 52—56.

[3] A. Lotze: History and development of grading theory. AEÜ *25* (1971), 9/10, 402—410.

[4] K. Kümmerle: Berechnungsverfahren für mehrstufige Koppelanordnungen mit konjugierter Durchschaltung. Systematik und Analyse. 9. Bericht über verkehrstheoretische Arbeiten, Institut für Nachrichtenvermittlung und Datenverarbeitung Universität Stuttgart 1969.

[5] K. Kümmerle: An analysis of loss approximations for link systems. AEÜ *25* (1971), 9/10, 466—471.

[6] J. N. Bridgford: The geometric group concept and its application to the dimensioning of link access systems. Fourth International Teletraffic Congress London 1964, Doc. 13.

[7] B. Wallström: Survey of "Congestion studies in telephone systems with overflow facilities". In: Prebook of the Fifth International Teletraffic Congress, New York 1967, 390—397.

[8] U. Herzog: A general variance theory applied to link systems with alternate routing. In: Prebook of the Fifth International Teletraffic Congress, New York 1967, 398—406.

[9] U. Herzog: Die Bemessung von Linksystemen mit alternativer Leitweglenkung nach dem verallgemeinerten RDA-Streuwertverfahren. Nachrichtentech. Z. *23* (1970), 3, 143—150.

[10] G. P. Basharin: Derivation of equations of state for two-stage telephone circuits with losses. Telecommunications (1960), 1, 79—90.

[11] R. Syski: Introduction to congestion theory in telephone systems. Oliver and Boyd, Edinburgh and London 1960.

[12] O. Neff: Programme zur exakten Berechnung von Linksystemen mit internem Überlauf. Monographie des Instituts für Nachrichtenvermittlung und Datenverarbeitung Universität Stuttgart 1971.

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