SAMPLED QUEUING SYSTEMS

M. Langenbach-Belz

Institute for Switching and Data Technics, University of Stuttgart, Stuttgart, GERMANY

Two models of sampled queuing systems are treated, one with batch service in fixed intervals and another one with batch arrivals in fixed intervals. For each of these two models the characteristic traffic values, i.e., probabilities of state, mean queue length, mean waiting time, probability of waiting, and waiting time distribution function are derived. Exact and approximate numerical values are given as a support for practical applications.

I. INTRODUCTION

I.1 Statement of the Problem

Present and future application of computers in communication networks and modern telephone and data switching systems require the analysis of sampled queuing systems. The basic structure of such sampled queuing systems is shown in Figure 1. The peripheral

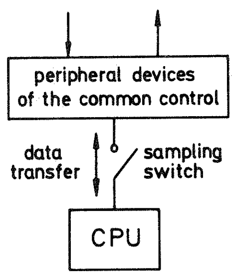

Fig. 1. Basic structure of sampled queuing systems.

devices correspond, e.g., to a common controlled telephone switching system, to a time sharing computer, or to data transmission devices. The data transfer between the peripheral devices and the Central Processing Unit (CPU) takes place in fixed intervals of time, caused by the periodically closed sampling switch. Hence, two basic subsystems can be distinguished from Figure 1. In the first subsystem corresponding to the peripheral devices, the preprocessed or waiting calls are transferred to the CPU in fixed intervals of time. This subsystem is named model A in this paper. In the second subsystem corresponding to the CPU, the arrivals of the calls take place in fixed intervals of time. This subsystem is named model B in this paper.

Presented at the Symposium on Computer-Communications Networks and Teletraffic
Polytechnic Institute of Brooklyn, April 4–6, 1972.

There are many types of each subsystem, regarding its input and/or service process, respectively, internal structure, queuing policy, etc. Some of them are dealt with in literature. Very often generating functions of probabilities of state or Laplace-Stieltjes transforms of waiting time distribution functions are given as results. In this paper new and explicit solutions are derived which can be applied directly to computer network design.

I.2 Description of the Two Models

2.1 Model A

This model corresponds to a peripheral device of a common controlled telephone or data switching system or to the queuing users of a time sharing computer, etc.

The calls arrive according to a Poisson process with the arrival rate λ. They have to wait in the queue in the order of their arrival (probability of waiting = 1). Always after a fixed interval of time T the sampling switch is closed and the waiting calls situated in the *n transfer places* in front of the queue are transferred, e.g., to the buffer of a CPU or to a service unit with a service time $\leq T$ (*batch service*). It is assumed, that this transfer cannot be blocked.

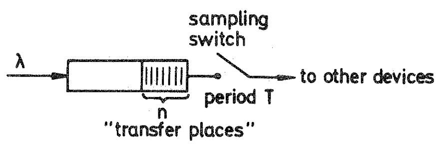

Fig. 2. Configuration of model A. The queue has an infinite number of waiting places.

This model is similar to the model of Bailey [3]. But contrary to Bailey, in this paper the state of the system is considered at an arbitrary instant t_v before the sampling clock. Moreover, additional results are derived.

2.2 Model B

The central service unit has a constant service time h. Always after a fixed period of c (c = integral number ≥ 1) service times the g peripheral devices are sampled synchronously. One peripheral device corresponds, e.g., to a system similar to model A. From each peripheral device i there are transferred $k = 0, 1, 2, \cdots, n_i$ calls per sampling clock with certain probabilities $q_i(k)$. It is assumed that the probabilities $q_i(k)$ are independent of each other and from themselves in successive instants of the sampling clock. The global probability of k arriving calls from all g devices is $r(k)$ with $k = 0, 1, 2, \cdots, m$ (*batch arrivals*).

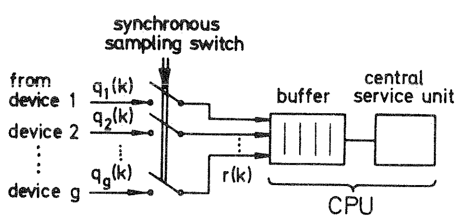

Fig. 3. Configuration of model B. The buffer has an infinite number of waiting places.

Within one global arriving batch the calls of device 2 are filed in the buffer behind the calls of device 1, the calls of device 3 are filed behind the calls of device 2, etc. The service discipline is first-come, first-served.

Besides the exact solution for an integral value of c in Section III, an approximation for the mean waiting time is dealt with in Section IV.2 for the case that $c = T/h$ is not an integral value but a real value.

II. ANALYSIS OF MODEL A

II.1 Generating Function of the Probabilities of State at an Arbitrary Instant

In statistical equilibrium the system is considered always at fixed instants t_v before the next sampling clock (see Figure 4). The value of t_v can be chosen arbitrarily between

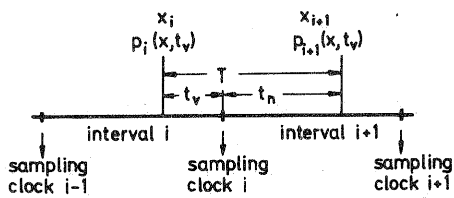

Fig. 4. Time intervals and sampling clock in model A.

0 and T. From Fig. 4 it can be seen that

$$T = t_v + t_n \quad . \tag{1}$$

The probability of x_i or x_{i+1} calls in the queue at an instant t_v before the sampling clock i or $i + 1$ shall be $p_i(x, t_v)$ or $p_{i+1}(x, t_v)$, respectively. The random variables x form an imbedded Markov chain. The number of arriving calls during the time intervals t_v or t_n or T shall be k_v or k_n or k, respectively. Then, directly before the sampling clock i there are $x_i + k_v$ calls in the system. Because n calls can be removed by the sampling clock i at most, there are only max $[(x_i + k_v - n), 0]$ calls in the system directly after the sampling clock i, where

$$\max\,[(x_i + k_v - n), 0] = \begin{cases} x_i + k_v - n, & x_i + k_v \geq n \ , \\ 0, & x_i + k_v < n \ . \end{cases}$$

The random variable x_{i+1} then is

$$x_{i+1} = \max\,[(x_i + k_v - n), 0] + k_n \quad . \tag{2}$$

The sum of the independent random variables x_i and k_v shall be the new random variable u_i

$$u_i = x_i + k_v \quad . \tag{3a}$$

With the abbreviation

$$v_i = \max\,[(u_i - n), 0] \,, \tag{3b}$$

Eq. (2) yields

$$x_{i+1} = v_i + k_n \,. \tag{4}$$

To derive the wanted probabilities of state $p(x, t_v)$ it is useful to introduce the generating function $Gx_{i+1}(z, t_v)$ of the probabilities $p_{i+1}(x, t_v)$. It is defined as

$$Gx_{i+1}(z, t_v) = \sum_{x=0}^{\infty} p_{i+1}(x, t_v)\, z^x \,. \tag{5}$$

Analogous to Eq. (5), $Gv_i(z)$ and $Gk_n(z)$ are the generating functions of the probabilities for v_i and k_n, respectively. Because v_i and k_n are independent random variables, it follows from Eq. (4) and from probability theory that

$$Gx_{i+1}(z, t_v) = Gv_i(z) \cdot Gk_n(z) \,. \tag{6}$$

$Gv_i(z)$ can be expressed in terms of $p_i(x, t_v)$ and the probabilities of k_v (using Equations (3a) and (3b)). In statistical equilibrium the probabilities of state must be invariable with time

$$p_i(x, t_v) = p_{i+1}(x, t_v) = p(x, t_v) \,.$$

Therefore, in Eq. (6) the indices i and $i + 1$ can be omitted. After several transformations of Eq. (6) and the assumption of a Poisson input process the expression for $Gx(z, t_v)$ is

$$Gx(z, t_v) = \frac{\sum\limits_{u=0}^{n-1} p(u, t_v) \sum\limits_{v=0}^{n-1-u} \frac{(\lambda t_v)^v}{v!}\,(z^n - z^{u+v})}{z^n - e^{\lambda T(z-1)}}\; e^{\lambda(t_n z - T)} \,. \tag{7}$$

The generating function (Eq. (7)) is similar to the generating function, which was obtained by Crommelin and Pollaczek for a queuing system without a sampling clock and with n servers and constant holding time $(M/D/n)$ [8, 9].

In the numerator of Eq. (7) there are still n unknown probabilities $p(u, t_v)$ which are eliminated in the following way. It is known from Crommelin and Pollaczek, that the denominator of Eq. (7) has exactly n real and complex roots $z_0, z_1, z_2, \cdots, z_v, \cdots, z_{n-1}$ inside and on the unit circle, where $z_0 = 1$ is always a root. Because for $z \leq 1$ the generating function $Gx(z, t_v)$ is also ≤ 1, the polynomial in the numerator of Eq. (7) must have exactly the same n roots z_v as the denominator. Hence, Eq. (7) can be written as

$$Gx(z, t_v) = \frac{K(z-1)(z-z_1)(z-z_2)\cdots(z-z_v)\cdots(z-z_{n-1})}{z^n - e^{\lambda T(z-1)}}\; e^{\lambda(t_n z - T)} \,. \tag{8}$$

The unknown constant K is determined by setting $z = 1$ in Eq. (8) because $Gx(1, t_v) = 1$.

Then, the final expression for $Gx(z, t_v)$ is after some transformations

$$Gx(z, t_v) = \frac{n - \lambda T}{\prod_{v=1}^{n-1} (1 - z_v)} \; \frac{\prod_{v=0}^{n-1} (z - z_v)}{z^n e^{\lambda T(1-z)} - 1} \; e^{\lambda t_v (1-z)} , \tag{9}$$

(z_v = roots of the denominator of Eq. (8) inside the unit circle).

Directly before the sampling clock there is $t_v = 0$. For this special value of t_v Eq. (9) results in the generating function obtained by Bailey [3].

For the special case of $n = 1$ Eq. (9) reduces to

$$Gx(z, t_v) = (1 - \lambda T) \frac{z - 1}{z e^{\lambda T(1-z)} - 1} e^{\lambda t_v (1-z)} . \tag{10}$$

II.2 Mean Queue Length at an Arbitrary Instant and Mean Waiting Time

The mean queue length $E[x, t_v]$ at a certain arbitrary instant t_v before the next sampling clock can be easily obtained from the generating function $Gx(z, t_v)$

$$E[x, t_v] = \left. \frac{dGx(z, t_v)}{dz} \right|_{z=1} . \tag{11}$$

The derivation of Eq. (9) with respect to z is (after some transformations)

$$\frac{dGx(z, t_v)}{dz} = Gx(z, t_v) \left[\frac{1}{z - 1} + \sum_{v=1}^{n-1} \frac{1}{z - z_v} - \frac{z^n e^{\lambda T(1-z)} \left(\frac{n}{z} - \lambda T \right)}{z^n e^{\lambda T(1-z)} - 1} - \lambda t_v \right] .$$

From this equation follows for $z = 1$ after a determination of the limiting value by the aid of L'Hôpital's rule

$$E[x, t_v] = \sum_{v=1}^{n-1} \frac{1}{1 - z_v} + \frac{(n - \lambda T)^2 - n}{2(\lambda T - n)} - \lambda t_v . \tag{12}$$

For the special case of $n = 1$, Eq. (12) reduces to

$$E[x, t_v] = \frac{1 - (1 - \lambda T)^2}{2(1 - \lambda T)} - \lambda t_v . \tag{13}$$

Again for the special case of $t_v = 0$, Eq. (12) becomes identical with that obtained by Bailey.

The mean queue length Ω referred to the whole sampling clock interval T is

$$\Omega = \frac{1}{T} \int_{t_v=0}^{T} E[x, t_v] \, dt_v ,$$

or with Eq. (12)

$$\Omega = \sum_{v=1}^{n-1} \frac{1}{1 - z_v} + \frac{(n - \lambda T)^2 - n}{2(\lambda T - n)} - \frac{1}{2} \lambda T . \tag{14}$$

On the other hand, Eq. (15) holds true generally

$$\Omega = \lambda t_w . \tag{15}$$

Therefore, the mean waiting time t_w is obtained from Eq. (14) and Eq. (15)

$$t_w = \frac{1}{\lambda}\left[\sum_{\nu=1}^{n-1} \frac{1}{1 - z_\nu} + \frac{n}{2}\left(\frac{1}{n - \lambda T} - 1\right)\right] . \tag{16}$$

Especially for $n = 1$

$$t_w = \frac{T}{2\,(1 - \lambda T)} . \tag{17}$$

It should be noted, that Eq. (17) is identical with the waiting time referred to the *waiting* calls in the system $M/D/1$. Furthermore t_w according to Eq. (16) is greater by $T/2$ than the mean waiting time referred to *all* calls in the system $M/D/n$. Figure 5 shows some numerical results for t_w.

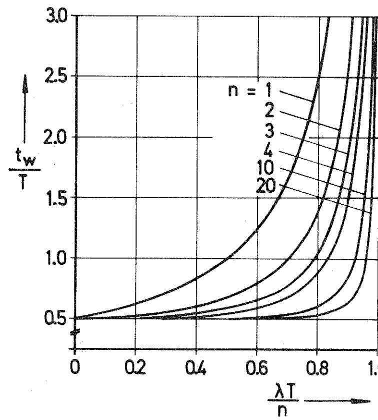

Fig. 5. Mean waiting time in model *A*.

II.3 Probabilities of State at an Arbitrary Instant

Probabilities of state $p\,(x, t_\upsilon)$ can be obtained by a retransformation of Equation (9). Generally this retransformation is

$$p\,(x, t_\upsilon) = \frac{1}{x!}\,\frac{d^x Gx\,(z, t_\upsilon)}{dz^x}\Bigg|_{z=0} ,$$

but unfortunately, the derivation of order x of the generating function cannot be given explicitly. Therefore, the generating function (Eq. (9)) is developed step by step into a serial expansion with respect to z. Then a comparison between the coefficients of this serial expansion and the coefficients of the general definition of the generating function (see Eq. (5)) yields the probabilities $p\,(x, t_\upsilon)$.

Equation (9) written in a different form

$$Gx\,(z, t_v) = \frac{\lambda T - n}{\prod_{v=1}^{n-1} (1 - z_v)} \; e^{\lambda t_v} \; \frac{\prod_{v=0}^{n-1} (z - z_v)}{1 - z^n e^{\lambda T(1-z)}} \; e^{-\lambda t_v z} \;, \tag{18}$$

with the serial expansion $\frac{1}{1 - a} = \sum_{\mu=0}^{\infty} a^{\mu}$ ($|a| < 1$), Eq. (18) can be written as

$$Gx\,(z, t_v) = \frac{\lambda T - n}{\prod_{v=1}^{n-1} (1 - z_v)} \; e^{\lambda t_v} \prod_{v=0}^{n-1} (z - z_v) \sum_{\mu=0}^{\infty} [z^n e^{\lambda T(1-z)}]^{\mu} \; e^{-\lambda t_v z} \;,$$

or

$$Gx\,(z, t_v) = \frac{\lambda T - n}{\prod_{v=1}^{n-1} (1 - z_v)} \; e^{\lambda t_v} \prod_{v=0}^{n-1} (z - z_v) \sum_{\mu=0}^{\infty} [z^{\mu n} e^{\mu \lambda T} e^{-\lambda z (\mu T + t_v)}] \;.$$

For the last exponential term in this expression the serial expansion $e^a = \sum_{j=0}^{\infty} \frac{a^j}{j!}$ is used

$$Gx\,(z, t_v) = \frac{\lambda T - n}{\prod_{v=1}^{n-1} (1 - z_v)} \; e^{\lambda t_v} \prod_{v=0}^{n-1} (z - z_v) \sum_{\mu=0}^{\infty} \left[z^{\mu n} e^{\mu \lambda T} \sum_{j=0}^{\infty} \frac{[-\lambda z\, (\mu T + t_v)]^j}{j!} \right] . \tag{19}$$

Equation (19) is a polynominal in z. Now this expression must be arranged with respect to powers of z to get the probabilities $p\,(x, t_v)$ as the coefficients of this polynomial. After some transformations this gives the explicit solution

$$p\,(x, t_v) = \frac{\lambda T - n}{\prod_{v=1}^{n-1} (1 - z_v)} \; e^{\lambda t_v} \left[\sum_{j=0}^{b-1} \left(e^{j \lambda T} \sum_{\mu=0}^{n} (-1)^{n-\mu} S_{n-\mu} \frac{[-\lambda\, (jT + t_v)]^{x - jn - \mu}}{(x - jn - \mu)!} \right) \right.$$

$$\left. + \; e^{b \lambda T} \sum_{\mu=0}^{x - bn} (-1)^{n-\mu} S_{n-\mu} \frac{[-\lambda\, (bT + t_v)]^{x - bn - \mu}}{(x - bn - \mu)!} \right] , \tag{20a}$$

for $bn \leq x < (b + 1)n$ $\quad (b = 0, 1, 2, \cdots)$ with

$$\left.\begin{aligned}
S_0 &= 1 \, , \\
S_1 &= (z_0 + z_1 + z_2 + \cdots + z_{n-1}) = \sum_{i=0}^{n-1} z_i \, , \\
S_2 &= (z_0 z_1 + z_0 z_2 + z_0 z_3 + \cdots + z_{n-2} z_{n-1}) = \sum_{\substack{i1,i2=0 \\ i1<i2}}^{n-1} z_{i1} z_{i2} \, , \\
&\vdots \\
S_n &= z_0 z_1 z_2 \cdots z_{n-1} \, ,
\end{aligned}\right\} \tag{20b}$$

and with

$$\sum_{i}^{j} (\cdot) = 0 \, (j < i). \tag{20c}$$

For the special case of $n = 1$, Eq. (20a) reduces to

$$p\,(x, t_v) = (1 - \lambda T)\, e^{\lambda t_v} \left[e^{x \lambda T} - \sum_{j=0}^{x-1} e^{j \lambda T} \frac{[-\lambda\,(jT + t_v)]^{(x-1-j)}}{(x - 1 - j)!} \left(1 + \frac{\lambda\,(jT + t_v)}{x - j}\right)\right]. \tag{21}$$

11.4 Waiting Time Distribution Function

The waiting time of a *test-call* is considered in order to determine the waiting time distribution function. If the test-call arrives at the instant t_v before the next sampling clock and finds

$0, 1, 2, \cdots, n - 1$ waiting calls in the system, it has to wait the time t_v ,

$n, n + 1, n + 2, \cdots, 2n - 1$ waiting calls in the system, it has to wait the time

$$T + t_v \, ,$$

$wn, wn + 1, wn + 2, \cdots, (w + 1)\,n - 1$ waiting calls in the system, it has to wait the time

$$wT + t_v \, .$$

The probability, that a call arrives during the interval dt_v at the instant t_v is constant and independent of t_v because of the Poisson arrival process. Thus, on the other hand the probability that the arriving test-call falls in the interval dt_v at the instant t_v is $\frac{dt_v}{T}$.

The probability that at the instant t_v there are already x calls waiting in front of the test-call is $p\,(x, t_v)$. Because these two probabilities are independent of each other, the probability $W\,(wT + t_v - dt_v, wT + t_v)$ that the test-call has to wait some time between $wT + t_v - dt_v$ and $wT + t_v$ is

$$W(wT + t_v - dt_v, wT + t_v) = \sum_{x=wn}^{(w+1)n-1} p(x, t_v) \frac{dt_v}{T}, \tag{22}$$

with $w = 0, 1, 2, \cdots$ and $0 \leq t_v \leq T$.

From Eq. (22), it follows with $dt_v \longrightarrow 0$ for the waiting time distribution function $P(\leq wT + t_v)$

$$P(\leq wT + t_v) = \frac{1}{T} \int_0^{t_v} \sum_{x=wn}^{(w+1)n-1} p(x, t_v)\, dt_v + P(\leq wT), \tag{23a}$$

with

$$P(\leq wT) = \sum_{v=0}^{w-1} \frac{1}{T} \int_0^{T} \sum_{x=vn}^{(v+1)n-1} p(x, t_v)\, dt_v . \tag{23b}$$

Equations (23a) and (23b) contain integrals I of the form

$$I = \frac{1}{T} \int_0^{t} \sum_{x=wn}^{(w+1)n-1} p(x, t_v)\, dt_v ,$$

which can be calculated by inserting Equation (20a).

As result for the waiting time distribution function one obtains (after several transformations) the explicit formula

$$P(\leq wT + t_v) = \frac{\lambda T - n}{\prod_{v=1}^{n-1}(1 - z_v)} \frac{1}{\lambda T} \sum_{\mu=0}^{n} (-1)^{n-\mu} S_{n-\mu} \left[\sum_{j=0}^{w-1} e^{\lambda(jT+t_v)} \left(\sum_{x=wn}^{(w+1)n-1} [(w+1)n - x] \frac{[-\lambda(jT+t_v)]^{x-jn-\mu}}{(x-jn-\mu)!} + n \sum_{f=1}^{(w-j)n-\mu} \frac{[-\lambda(jT+t_v)]^{(w-j)n-\mu-f}}{[(w-j)n-\mu-f]!} \right) + e^{\lambda(wT+t_v)} \sum_{x=0}^{n-1-\mu} (n - x - \mu) \frac{[-\lambda(wT+t_v)]^x}{x!} - [(w+1)n - \mu] \right], \tag{24}$$

with

$$w = 0, 1, 2, \cdots \text{ and } 0 \leq t_v \leq T ,$$

and additionally Equations (20b) and (20c).

For the special case of $n = 1$, Eq. (24) reduces to

$$P(\leq wT + t_v) = \frac{1 - \lambda T}{\lambda T} \left[\sum_{j=0}^{w} \frac{[-\lambda(jT+t_v)]^{w-j}}{(w-j)!} e^{\lambda(jT+t_v)} - 1 \right]. \tag{25}$$

It should be noted that Eq. (25) is identical with the waiting time distribution function referred to the *waiting* calls in the system $M/D/1$.

The complementary waiting time distribution function $P(> wT + t_v)$ is

$$P(>wT+t_v) = 1 - P(\leq wT+t_v) \ . \tag{26}$$

An example for $P(>wT+t_v)$ is given in Figure 6.

II.5 Mutual Influence of the Parameters n and T on the Mean Waiting Time

In many computer controlled queuing systems the performance of the sampling is correlated with an interrupt in the central processor unit. These interrupts cause an additional load for the central processor unit. Therefore, from the point of view of the processor unit, it is desirable to keep the number of interrupts per time unit as small as possible, i.e., to make the sampling clock interval T as long as possible. But on the other hand, the absolute mean waiting time of a call in the queue must not be increased too much.

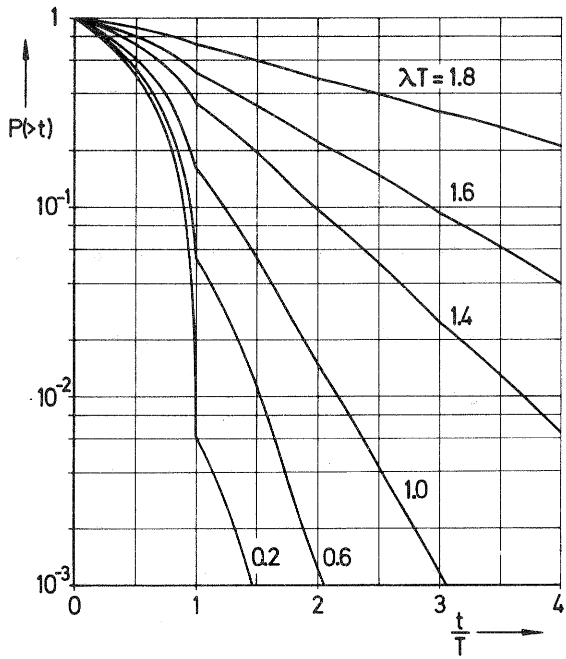

Fig. 6. Complementary waiting time distribution in model A for the case of $n = 2$.

To study the problem mentioned above, the absolute mean waiting time is considered in the case of an increased sampling clock interval T. But in order to keep the mean waiting time under a certain limit, also the number of transfer places is increased such that n/T is constant. Figure 7 shows the result of these considerations. This diagram can be interpreted in the following way: For small arrival rates λ it is not recommendable to increase T and n because the absolute mean waiting time is also increased quite notably. Furthermore, for small λ the additional load for the processor caused by the interrupts does not matter. However, for large λ the load of the processor is so great, that the additional load caused by the interrupts is quite notable. In this case it is worth increasing n and T, because the absolute mean waiting time remains in the same range though the number of interrupts is considerably decreased.

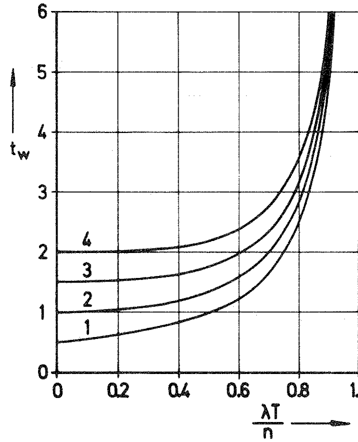

Fig. 7. Mutual influence of n and T on the absolute mean waiting time in model A. T and t_w are in arbitrary time units. Curve 1: $T = 1, n = 1$. Curve 2: $T = 2, n = 2$, Curve 3: $T = 3, n = 3$. Curve 4: $T = 4, n = 4$.

II.6 Probabilities for the Number of Transferred Calls per Sampling Clock

If there are $x < n$ waiting calls in the system directly before the sampling clock, all x calls are removed from the queue and are transferred to the following device. If there are $x \geq n$ waiting calls in the system, n calls are transferred. Therefore, the probabilities $q(i)$ of i transferred calls can be easily obtained from Eq. (20a) with $t_v = 0$

$$q(i) = p(i, 0) \qquad (0 \leq i < n) ,$$

$$q(n) = \sum_{x=n}^{\infty} p(x, 0) = 1 - \sum_{x=0}^{n-1} p(x, 0) = 1 - \sum_{x=0}^{n-1} q(x) . \tag{27}$$

III. ANALYSIS OF MODEL B

III.1 Generating Function of the Probabilities of State Directly Before and Directly After the Sampling Clock

At first, the system (see Fig. 3) is considered with the global arrival group directly after the sampling clock (see Figure 8).

The probability of x_i or x_{i+1} calls (including a call in the service unit) directly after the sampling clock i or $i + 1$ be $p_{n,i}(x)$ or $p_{n,i+1}(x)$, respectively. The random variables x form and imbedded Markov chain. The probability of k_i or k_{i+1} calls (with $k = 0, 1, 2, \cdots, m; m > c$) arriving with the sampling clock i or $i + 1$ be $r_i(k)$ or $r_{i+1}(k)$, respectively.

During the period T between two sampling clocks there can only be served c calls at most. Therefore, the random variable x_{i+1} is

$$x_{i+1} = \max [(x_i - c), 0] + k_{i+1} , \tag{28}$$

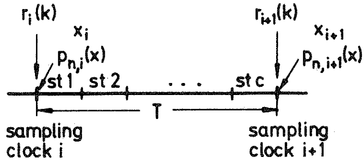

Fig. 8. Time intervals and sampling clock in model *B*. (*st* = service time)

with

$$\text{Max}\ [(x_i - c), 0] = \begin{cases} x_i - c, & x_i \geq c\ , \\ 0, & x_i < c\ . \end{cases}$$

With the abbreviation

$$v_i = \text{Max}\ [(x_i - c), 0]\ , \tag{29}$$

Eq. (28) can be written as

$$x_{i+1} = v_i + k_{i+1}\ . \tag{30}$$

Equations (28), (29), and (30) are similar to Eqs. (2), (3b) and (4) of model *A*. So the generating function $Gx_n(z)$ of the probabilities of state $p_n(x)$ in the case of statistical equilibrium can be obtained analogously to Eq. (7)

$$Gx_n(z) = \frac{\sum_{i=0}^{c-1} p_n(i)(z^c - z^i)}{z^c - Gk(z)} Gk(z)\ . \tag{31a}$$

Where

$$Gk(z) = \sum_{k=0}^{m} r(k) z^k\ , \tag{31b}$$

is the generating function of the probabilities $r(k)$.

It can be shown, that the denominator of Eq. (31a) has exactly c roots (real and complex) $z_0, z_1, z_2, \cdots, z_\nu, \cdots, z_{c-1}$ inside and on the unit circle, where $z_0 = 1$ is always a root. Therefore, the unknown probabilities $p_n(i)$ in the numerator of Eq. (31a) are eliminated in the same manner as in Equation (7). The result is analogous to Eq. (9)

$$Gx_n(z) = \frac{c - E[k]}{\prod_{\nu=1}^{c-1}(1 - z_\nu)} \frac{\prod_{\nu=0}^{c-1}(z - z_\nu)}{z^c - Gk(z)} Gk(z)\ , \tag{32}$$

where

$$E[k] = \sum_{k=0}^{m} k\, r(k) = \text{expectation of } k\ . \tag{33}$$

Another expression for $Gx_n(z)$ is obtained, when the c factors $(z - z_\nu)$ with $|z_\nu| \leq 1$ are cancelled out of the numerator and the denominator of Equation (31a). Then only $m - c$ roots $z_c, z_{c+1}, z_{c+2}, \cdots, z_\nu, \cdots, z_{m-1}$ with $|z_\nu| > 1$ remain in the expression for $Gx_n(z)$

$$Gx_n(z) = \prod_{\nu=c}^{m-1} \frac{1 - z_\nu}{z - z_\nu} Gk(z) \, . \tag{34}$$

If $c << m - c$ it is suitable to use Eq. (32), if $c >> m - c$ it is suitable to use Equation (34).

The number x_n of calls in the system (buffer + server) directly after the sampling clock is the sum of the number x_ν of calls in the system directly before the sampling clock and the number k of calls arriving with the sampling clock. Because the latter two are independent random variables, the generating function $Gx_\nu(z)$ of the probabilities of state $p_\nu(x)$ directly before the sampling clock may be calculated from the equation

$$Gx_n(z) = Gx_\nu(z)\, Gk(z) \, .$$

Applying Eq. (34) to this formula leads to

$$Gx_\nu(z) = \frac{Gx_n(z)}{Gk(z)} = \prod_{\nu=c}^{m-1} \frac{1 - z_\nu}{z - z_\nu} \, . \tag{35}$$

III.2 Mean Queue Length

The mean queue length $E_\nu[x]$ directly before the sampling clock is

$$E_\nu[x] = \left. \frac{dGx_\nu(z)}{dz} \right|_{z=1} \, .$$

The derivation of Eq. (35) with respect to z is after some transformations

$$\frac{dGx_\nu(z)}{dz} = \prod_{\nu=c}^{m-1} \frac{1 - z_\nu}{z - z_\nu} \sum_{\mu=c}^{m-1} \frac{1}{z_\mu - z} \, . \tag{36}$$

For $z = 1$ Eq. (36) results in

$$E_\nu[x] = \sum_{\mu=c}^{m-1} \frac{1}{z_\mu - 1} \, . \tag{37}$$

The mean queue length $E_n[x]$ directly after the sampling clock then is

$$E_n[x] = E_\nu[x] + E[k] \, . \tag{38}$$

III.3 Mean Waiting Time

In front of an arriving group there are in the mean $E_\nu[x]$ calls in the system (including a call in the service unit). If at the moment of the group arrival, there is a call in the service unit, this call starts its service time at the same moment. Therefore if the group size of the arriving calls is k, the first call of this group has to wait in the mean

$E_v\,[x]$ service periods, the second call has to wait in the mean $E_v\,[x] + 1$ service periods, etc. and the last call has to wait in the mean $E_v\,[x] + k - 1$ service periods. So the mean waiting time $t_w^*(k)$ of all calls of this group of size k is

$$t_w^*(k) = \frac{1}{k}\,[E_v\,[x]\,h + (E_v\,[x] + 1)\,h + \cdots + (E_v\,[x] + k - 1)\,h] \;,$$

or

$$t_w^*(k) = \left(E_v\,[x] + \frac{k-1}{2}\right)h \;.$$

In the mean, $r(k)\,k$ calls arrive per sampling clock which will have the mean waiting time $t_w^*(k)$. Altogether $E\,[k]$ calls arrive per sampling clock in the mean. Then, the mean waiting time referred to all arriving calls is the weighted sum of the different $t_w^*(k)$ with $k = 1$ through m

$$t_w^* = \sum_{k=1}^{m} \frac{r(k)\cdot k}{E\,[k]}\, t_w^*(k) \;,$$

$$= \frac{h}{E\,[k]} \sum_{k=1}^{m} r(k)\,k \left(E_v\,[x] + \frac{k-1}{2}\right),$$

$$= E_v\,[x]\,h + \frac{\sum_{k=1}^{m} r(k)\,k\,(k-1)}{2E\,[k]}\, h \;, \qquad (39a)$$

or

$$t_w^* = E_v\,[x]\,h + \frac{1}{2}\left(\frac{\mathrm{Var}\,[k]}{E\,[k]} + E\,[k] - 1\right)h \;, \qquad (39b)$$

with $E_v\,[x]$ from Eq. (37) and $\mathrm{Var}\,[k] = \sum_{k=0}^{m} (k - E\,[k])^2\, r(k)$ = variance of k. For the special case of $c = 1$ Eq. (39b) reduces to

$$t_w^* = \frac{1}{2}\left(\frac{\mathrm{Var}\,[k]}{E\,[k]}\; \frac{1}{1 - E\,[k]} - 1\right)h \;.$$

III.4 Probability of Waiting

First, the probability $P\,(=0)$ is calculated, that an arriving call does not have to wait. Only the first call of an arriving group needs not wait when the system is empty. The probability that the system is empty directly before the sampling clock is $p_v(0)$. $p_v(0)$ can be obtained from Eq. (35) by setting $z = 0$

$$p_v(0) \equiv Gx_v(0) = \prod_{\nu=c}^{m-1} \frac{z_\nu - 1}{z_\nu} \;. \qquad (40)$$

The probability that a call arrives at all with the sampling clock is $1 - r(0)$. Thus, in the mean $p_v(0)(1 - r(0)) \cdot 1$ calls arrive per sampling clock which do not have to wait.

Altogether $E[k]$ calls arrive per sampling clock in the mean. Therefore, the probability $P(=0)$ is

$$P(=0) = \frac{p_v(0)(1 - r(0))}{E[k]} ,$$

or with Eq. (40)

$$P(=0) = \frac{1 - r(0)}{E[k]} \prod_{v=c}^{m-1} \frac{z_v - 1}{z_v} . \tag{41}$$

The probability of waiting $P(>0)$ then is

$$P(>0) = 1 - \frac{1 - r(0)}{E[k]} \prod_{v=c}^{m-1} \frac{z_v - 1}{z_v} . \tag{42}$$

For the mean waiting time t_w referred to the waiting calls, the following equation holds true

$$t_w = \frac{t_w^*}{P(>0)} , \tag{43}$$

with t_w^* from Eq. (39b) and $P(>0)$ from Equation (42).

III.5 Probabilities of State $p_v(x)$

The generating function (Eq. (35)) can be decomposed into partial fractions

$$Gx_v(z) = \sum_{i=c}^{m-1} \frac{\prod_{v=c}^{m-1} (1 - z_v)}{\prod_{\substack{v=c \\ v \neq i}}^{m-1} (z_i - z_v)} \frac{1}{(z - z_i)} = - \sum_{i=c}^{m-1} \frac{\prod_{v=c}^{m-1} (1 - z_v)}{z_i \prod_{\substack{v=c \\ v \neq i}}^{m-1} (z_i - z_v)} \frac{1}{\left(1 - \frac{z}{z_i}\right)} .$$

The last term of this expression is expanded into a series

$$Gx_v(z) = - \sum_{i=c}^{m-1} \left[\frac{\prod_{v=c}^{m-1} (1 - z_v)}{z_i \prod_{\substack{v=c \\ v \neq i}}^{m-1} (z_i - z_v)} \sum_{x=0}^{\infty} \left(\frac{z}{z_i}\right)^x \right] . \tag{44}$$

Equation (44) is a polynomial in z. The coefficients of this polynomial are the wanted probabilities of state $p_v(x)$

$$p_v(x) = -\sum_{i=c}^{m-1} \left[\frac{\prod_{v=c}^{m-1} (1 - z_v)}{\prod_{\substack{v=c \\ v \neq i}}^{m-1} (z_i - z_v)} \frac{1}{z_i^{x+1}} \right] . \tag{45}$$

III.6 Waiting Time Distribution Function

Because the waiting time of the calls is always an integral multiple of the holding time h, the waiting time distribution function is a step function. Therefore, the waiting time distribution function is represented here as $P(\leq dh)$, where d is an integral number ≥ 0. The probability that a call has to wait exactly the time interval dh shall be $W(d)$. The probabilities $W(d)$ are calculated analagously to the probability of waiting in Section III.4.

The derivation of the probabilities $W(d)$ is shown by example of $W(1)$. A call has to wait for one service period when:

(1) Either it is the first call of the arriving group and there is one call in front of the arriving group
(2) Or it is the second call of the arriving group and there are no calls in front of the arriving group.

The probability, that there is no or one call in front of an arriving group, is $p_v(0)$ or $p_v(1)$, respectively. The probability, that the arriving group contains at least 1 or 2 calls, is $1 - r(0)$ or $1 - r(0) - r(1)$, respectively. Therefore with each sampling clock

$$p_v(0)\,[1 - r(0) - r(1)] \cdot 1 + p_v(1)\,[1 - r(0)] \cdot 1 ,$$

calls arrive in the mean, which will have a waiting time of one service period. Altogether $E[k]$ calls arrive in the mean per sampling clock. Thus

$$W(1) = \frac{1}{E[k]} \left[p_v(0)\,[1 - r(0) - r(1)] + p_v(1)\,[1 - r(0)]\right] . \tag{46}$$

Analogously to $W(1)$ the general expressions for $W(d)$ are derived

$$W(d) = \begin{cases} \dfrac{1}{E[k]} \cdot \displaystyle\sum_{x=0}^{d} p_v(x) \left[1 - \sum_{k=0}^{d-x} r(k)\right], & d < m , \\ \dfrac{1}{E[k]} \cdot \displaystyle\sum_{x=d-m+1}^{d} p_v(x) \left[1 - \sum_{k=0}^{d-x} r(k)\right], & d \geq m , \end{cases} \tag{47}$$

with $p_v(x)$ from Equation (45). The waiting time distribution function $P(\leq dh)$ is

$$P(\leq dh) = \sum_{j=0}^{d} W(j) ,$$

or with Eq. (47)

$$P(\leq dh) = \begin{cases} \dfrac{1}{E[k]} \displaystyle\sum_{j=0}^{d} \sum_{x=0}^{j} p_v(x) \left[1 - \sum_{k=0}^{j-x} r(k)\right], & d < m , \\ P(\leq (m-1)h) + \dfrac{1}{E[k]} \displaystyle\sum_{j=m}^{d} \sum_{x=j-m+1}^{j} p_v(x) \left[1 - \sum_{k=0}^{j-x} r(k)\right], & d \geq m . \end{cases} \tag{48}$$

The complementary waiting time distribution function is

$$P(>dh) = 1 - P(\leq dh) .$$

The complementary waiting time distribution function $W(>dh)$ referred to the waiting calls is

$$W(>dh) = \frac{P(>dh)}{P(>0)} .$$

III.7 Distinction of Calls Coming from Different Peripheral Devices

As is shown in Fig. 3, the calls may come from g different peripheral devices with the probability of group size $q_i(k)$. Since now only the global arriving group with probability of group size $r(k)$ was regarded, where $r(k)$ is the convolution of the probabilities $q_i(k)$, it is possible to calculate all quantities, which were derived in the Sections III.1–III.6 for the calls of the global arrival group and also for the calls coming from any peripheral device i.

The mean queue size in front of the arriving calls of device i shall be $E_{v,i}[x]$. Then $E_{v,1}[x]$ is identical to $E_v[x]$ of Equation (37). Because all calls of the devices $< i$ are arranged before the calls of device i in the global arriving group, $E_{v,i}[x]$ is

$$E_{v,i}[x] = E_{v,1}[x] + \sum_{j=1}^{i-1} E[k_j] \quad (i > 1) , \tag{49}$$

where $E[k_i] = \sum_{k=0}^{n_i} k\, q_i(k)$.

Analogous to the mean waiting time of all calls of the global group in Eq. (39b), the mean waiting time $t^*_{w,i}$ of all calls coming from device i is

$$t^*_{w,i} = E_{v,i}[x]\, h + \frac{1}{2}\left(\frac{\mathrm{Var}[k_i]}{E[k_i]} + E[k_i] - 1\right) h . \tag{50}$$

In front of the arriving calls of device i there are no further calls, if the system is empty at the arrival moment of the global group and if the global group contains no calls of devices $< i$. The probability $p_{v,i}(0)$, that the arriving calls of device i find the system empty, is therefore

$$p_{v,i}(0) = \begin{cases} p_v(0)\,, & i = 1\,, \quad (51a) \\ p_v(0) \prod_{j=1}^{i-1} q_j(0)\,, & i > 1\,, \quad (51b) \end{cases}$$

where $p_v(0)$ is identical to Equation (40).

Then, analogous to the probability of waiting for the calls of the global arriving group in Eq. (42), the probability of waiting $P_i(>0)$ for the calls of device i is

$$P_i(>0) = 1 - \frac{1 - q_i(0)}{E[k_i]}\, p_{v,i}(0)\,. \quad (52)$$

The probabilities $p_{v,i}(x)$, that there are x calls in front of the arriving calls of device i can be obtained by a convolution of the probabilities $p_v(x)$ of Eq. (45) and the probabilities $q_1(k)$ through $q_{i-1}(k)$ (similar to $p_{v,i}(0)$).

Then, the waiting time distribution function $P_i(\leq dh)$ of all calls coming from device i is in analogy to Eq. (48)

$$P_i(\leq dh) = \begin{cases} \dfrac{1}{E[k_i]} \displaystyle\sum_{j=0}^{d} \sum_{x=0}^{j} p_{v,i}(x) \left[1 - \sum_{k=0}^{j-x} q_i(k)\right], & d < n_i\,, \\ P_i(\leq (n_i - 1)h) + \dfrac{1}{E[k_i]} \displaystyle\sum_{j=n_i}^{d} \sum_{x=j-n_i+1}^{j} p_{v,i}(x) \left[1 - \sum_{k=0}^{j-x} q_i(k)\right], & d \geq n_i\,. \end{cases} \quad (53)$$

The complementary waiting time distribution function $W_i(>dh)$ referred to the waiting calls of device i is

$$W_i(>dh) = \frac{(1 - P_i(\leq dh))}{P_i(>0)}\,.$$

IV. APPROXIMATIONS OF THE MEAN WAITING TIME OF MODEL B

IV.1 Diagrams for the Mean Waiting Time t_w^*

For practical applications (e.g., in network dimensioning) one should have in particular diagrams of the mean waiting time vs all interesting parameters such as the offered traffic (characterized by $E[k]$, $r(k)$, the maximum size m of the global arriving batch per sampling clock) and the maximum number of service periods c per clock interval. Because of this variety of parameters it is impossible to compress the most interesting parameter combinations in some few handy diagrams. The following method for approximate diagrams yields, however, values of t_w^* rather close to the exact ones.

The second term in Eq. (39b) depends only on Var $[k]$ and $E[k]$. Tests have shown, that the value of $E_v[x]$ has approximately the same magnitude for different combinations of m and probabilities $r(k)$, provided that $E[k]$ and Var $[k]$ are constant.

Therefore, diagrams for a fixed value c can be drawn with an approximate t_w^* as a function of $E\ [k]$, where Var $[k]/E\ [k]$ can be used as a parameter. Figure 9 shows an example for $c = 2$ and $m = 6$. This figure can be used also for $m \neq 6$. For $m < 6$ the exact values of t_w^* tend to be smaller and for $m > 6$ the exact values of t_w^* tend to be greater than the approximate values in these curves.

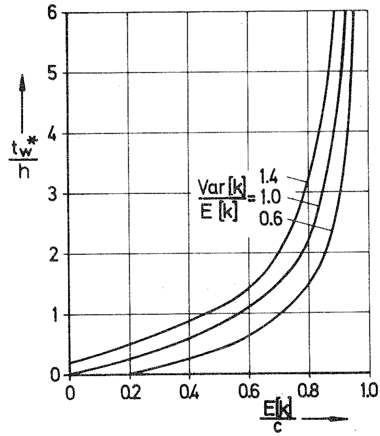

Fig. 9. Approximate mean waiting time in model B with parameter Var $[k]/E\ [k]$ for the case of $c = 2$ and $m = 6$.

IV.2 Approximation of the Mean Waiting Time t_w^* in a System Where the Clock Period is not an Integral Multiple of the Holding Time

If the sampling clock period is not an integral multiple of the holding time, the following approximations for the mean waiting time t_w^* are applicable: The sampling clock period shall be $T = ch$, where c is a real value between the two successive integral values c_u and c_0. Then, the mean waiting times t_{w,c_u}^* and t_{w,c_0}^* are calculated according to Eq. (39b) with the given probabilities $r\ (k)$ of the input process once with c_u and once with c_0, respectively. The mean waiting time t_w^* then is obtained approximately by linear interpolation between t_{w,c_u}^* and t_{w,c_0}^*

$$t_w^* = (c - c_u)\, t_{w,c_0}^* + (c_0 - c)\, t_{w,c_u}^* \ .$$

Of course, the same method can be used by means of the approximate diagrams described in Section IV.1. In this case the approximate mean waiting time t_w^* can be obtained by a corresponding linear interpolation between two curves belonging to c_u and c_0, the same value of the parameter Var $[k]/E\ [k]$ and the same abscissa value $E\ [k]/c$.

Simulation runs on digital computers have proved, that these handy methods yield sufficient values for practical applications.

V. CONCLUSION

Many specific problems of computer aided communication networks, multiaccess computers and other systems with common control can be analyzed by one of the models treated in this paper. Besides the theoretical results also exact and approximate numerical results presented here, are useful for practical engineering.

More complex models of sampled queuing systems are achieved by a combination of model A and model B (to be published at some future time).

REFERENCES

[1] B. A. Powell and B. Avi-Itzhak, "Queuing Systems with Enforced Idle Time," *Operations Res.*, **15**, 1145–1156 (1967).

[2] L. E. N. Delbrouck, "A Multiserver Queue with Enforced Idle Times," *Operations Res.*, **17**, 506–518 (1969).

[3] N. T. J. Bailey, "On Queuing Processes with Bulk Service," *J. Roy. Stat. Soc., Ser. B.*, **16**, 80–87 (1954).

[4] F. Downton, "Waiting Time in Bulk Service Queues," *J. Roy. Stat. Soc., Ser. B.*, **17**, 256–261 (1955).

[5] W–C Chan, "Computer-Controlled Queuing System with Constant-Access Cycle and General Service Times," *Proc. IEE*, **117**, 927–930 (May 1970).

[6] R. G. Miller, "A Contribution to the Theory of Bulk Queues," *J. Roy. Stat. Soc., Ser. B.*, **21**, 320–337 (1969).

[7] P. E. Boudreau and M. Kac, "Analysis of a Basic Queuing Problem Arising in Computer Systems," *IBM J. Res. Dev.*, **5**, 132–140 (April 1961).

[8] C. D. Crommelin, "Delay Probability Formulae When the Holding Times are Constant," *Post. Off. Elect. Engrs.*, **25**, 41–50 (1932). Also: **26**, 266–274 (1933/34).

[9] F. Pollaczek, "Zur Theorie des Wartens vor Schaltergruppen," *Elekt. Nachr. Techn.*, **9**, 434–454 (1932).

[10] W. Feller, "An Introduction to Probability Theory and Its Applications," **1** (New York: John Wiley 1957).