LOSS SYSTEMS WITH DISPLACING PRIORITIES

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ABSTRACT

In this paper a full available trunkgroup with n channels is considered, to which R Poisson input streams corresponding to R priority classes are offered. An arriving call, finding all channels occupied, can displace an established call of the momentarily lowest priority class, provided that this class is lower than its own class. Three disciplines how to select the established call to be displaced are considered: "first-come, last-displaced", "first-come, first-displaced" and "random-displacement".

At first the discipline-invariable probabilities of loss or displacement, resp., are derived.

Furthermore these probabilities will be determined in the case, where established calls are displaced by arriving calls not only of a higher but also of the same priority class.

For this case as well as for the disciplines "first-come, last-displaced" and "random-displacement" the following distribution functions are derived for each priority class:

- The distribution $P_x(>t)$ or $P_{x_1,x_2}(>t)$, resp., for a call, starting in the state x or x_1,x_2. resp.
- The distribution $P(>t)$ for an arbitrary call
- The distribution $P_d(>t)$ referred to displaced calls only
- The distribution $P_s(>t)$ referred to nondisplaced calls only

as well as the corresponding mean holding times.

1. INTRODUCTION

In non-public switching networks e.g. for flight-traffic control or weather-service it may be useful to assign to each call a certain priority class. If a call finds all channels of a trunk group busy, it will interrupt and displace one of the existing calls, provided that this established call is of lower priority. So an incoming call is only lost, if at the time of its arrival all channels are occupied by calls of a higher or its own priority.

This above mentioned priority assignment could also be useful in public communication networks in the case of cable breakdowns, catastrophes etc. By this application it would not be necessary to switch off all "non-important" subscribers from the exchange during the repair time or for the duration of the catastrophe.

In this paper a full available trunk group with n channels is considered. The following assumptions, concerning the offered traffic, will be made:

- R input streams - according to R priority classes - are offered to the trunk group
- Each input stream is Poissonian with the arrival rate λ_r ($r = 1,2, \ldots R$).
- All the calls have the same time-independent terminating rate $\mu_r = \mu$.
- Lost or displaced calls are cleared

The rules for the selection of the established call to be displaced constitute the displacing-discipline. Three disciplines will be considered: first-come, last-displaced; first-come, first-displaced and random-displacement. Of course an existing call of the momentarily lowest class, i.e. the class with the highest priority number, will be displaced.

Under the condition that only established calls of lower priorities can be displaced, the displacing-discipline has no influence on the probabilities of loss and displacement.

On the contrary, the holding time distribution of a certain priority class r ($r = 2,3, \ldots R$) will depend on these disciplines.

2. THE PROBABILITY OF LOSS, THE PROBABILITY OF DISPLACEMENT

These probabilities can be derived from the state probabilities of the system. To calculate these state probabilities, the following facts have to be taken into account:

a) The input and occupation processes of the classes r+1, r+2, ... R-1, R have no influence on those of class r ($r = 1,2, \ldots R-1$), where class 1 has the highest and class R the lowest priority.

b) As a consequence of a) the probability of the state "x_1 lines occupied by class 1" is given by an Erlang distribution /1/.

) If it is possible to prove that the probability of a state "(x_1+x_2) lines occupied by the classes 1 and 2 " is also an Erlang distribution, then we obviously obtain an Erlang distribution for the probability "$(x_1+x_2+ \dots +x_r)$ lines occupied by the classes 1,2, ... r".

herefore it is necessary to find the state proability p_x, where $x = x_1+x_2$. First we define the tate probability p_{x_1,x_2}, i.e. the probability or "x_1 lines occupied by class 1, x_2 lines occuied by class 2".

y using the statistical equilibrium we obtain or p_{x_1,x_2} the following set of equations:

$x<n$:

$$(\mu x+\lambda_1+\lambda_2)\cdot p_{x_1,x_2} = \mu(x_2+1)\cdot p_{x_1,x_2+1} +\mu(x_1+1)\cdot p_{x_1+1,x_2} +\lambda_2\cdot p_{x_1,x_2-1} +\lambda_1\cdot p_{x_1-1,x_2} \quad (1)$$

$x=n,\ x_2>0$:

$$(\mu x+\lambda_1)\cdot p_{x_1,x_2} = \lambda_1\cdot p_{x_1-1,x_2+1} +\lambda_2\cdot p_{x_1,x_2-1} +\lambda_1\cdot p_{x_1-1,x_2} \quad (2)$$

$x=n,\ x_2=0$:

$$\mu x\cdot p_{x_1,x_2} = \lambda_1\cdot p_{x_1-1,x_2+1} +\lambda_1\cdot p_{x_1-1,x_2} \quad (3)$$

e must replace nonexisting state probabilities y the value zero.

y introducing the offered traffic

$$A_r = \frac{\lambda_r}{\mu} \quad (4)$$

s well as $x_1 = x-x_2$ in eq.(1,2,3) and summaizing the eq.(1) and (2) from $x_2 = 0$ to $x_2 = x$ e obtain after some tedious transformations:

$x<n$:

$$(x+A_1+A_2)\cdot p_x = (x+1)\cdot p_{x+1} +(A_1+A_2)\cdot p_{x-1} \quad (5)$$

$x=n$:

$$n\cdot p_n = (A_1+A_2)\cdot p_{n-1} \quad (6)$$

f eq.(5) is arranged in the following way

$$(A_1+A_2)\cdot p_x -(x+1)\cdot p_{x+1} = (A_1+A_2)\cdot p_{x-1} -x\cdot p_x \quad (7)$$

t is obvious that (5) is fulfilled, if

$$(A_1+A_2)\cdot p_{x-1} -x\cdot p_x = c = \text{const.}$$

s a consequence of eq.(6) as a boundary condiion of (5) it follows that $c = 0$.

bviously the solution of eq.(6,7) is the rlang distribution with the offered traffic A_1+A_2).

his result can be easily interpreted: If an esablished 2-call with the time-independent terinating-rate μ is diplaced and the occupation s continued by a 1-call of the same terminatng-rate μ, then the occupation process is not nfluenced. The displacing, new 1-call has the ame holding time distribution exp(-ht) as the emaining part of the displaced call would have ad.

Therefore the state probability p_x of the state "$(x_1+x_2+ \dots x_r)$ lines occupied by calls of the classes 1,2, ... r" is

$$p_x = \frac{\dfrac{A_{\leq r}^x}{x!}}{\sum_{i=0}^{n} \dfrac{A_{\leq r}^i}{i!}} \quad (8)$$

where

$$A_{\leq r} = \sum_{i=1}^{r} A_i$$

This will facilitate the further derivations. Regarding class r, the input and occupation process of the classes 1,2, ... r-1 can be treated as one process resulting from one offered traffic $A_1+A_2+ \dots A_{r-1}$. So, henceforth, we will consider only two classes: Class 1 as the high priority class and class 2 as the low priority class, corresponding to the classes 1,2, ... r-1 and class r, resp. The calls of class 1 will be denoted for the sake of briefness with 1-calls, and analogously those of class 2 with 2-calls.

2.1 Class 1

The probability of loss B_1, the carried load Y_1 and the probability p_{x_1} for "x_1 lines occupied by class 1" are identical with the well known results found by A.K.Erlang /1/:

$$p_{x_1} = \frac{\dfrac{A_1^{x_1}}{x_1!}}{\sum_{i=0}^{n} \dfrac{A_1^i}{i!}} \quad (9)$$

$$B_1 = p_{x_1=n} = E_n(A_1) \quad (10)$$

$$Y_1 = \lambda_1(1-B_1)h_1 = A_1[1-B_1]\frac{h_1}{h} = A_1[1-E_n(A_1)] \quad (11)$$

2.2 Class 2

It holds (cf. eq.(8)):

$$p_{x_1+x_2} = \frac{\dfrac{(A_1+A_2)^{x_1+x_2}}{(x_1+x_2)!}}{\sum_{i=0}^{n} \dfrac{(A_1+A_2)^i}{i!}} \quad (12)$$

Therefore

$$B_2 = p_{x_1+x_2=n} = E_n(A_1+A_2) \quad (13)$$

The carried load is given by

$$Y_2 = \lambda_2(1-B_2)h_2 = A_2(1-B_2)\frac{h_2}{h} \quad (14)$$

The mean holding time h_2 of a 2-call is not yet known and will be determined later on (eq.(16)). As 2-calls can be displaced, the mean holding time will be less than h.

The expectation value of the holding time of a displacing call is $h = 1/\mu$ and the remaining part of the corresponding displaced call would have had the same expectation value h. So it is evident that the carried traffic of class 2 is reduced by displacements in the same amount as the carried traffic of class 1 increases. That means

$$\Delta Y_1 = -\Delta Y_2$$

For ΔY_1 holds

$$\Delta Y_1 = A_1[1-E_n(A_1)] -A_1[1-E_n(A_1+A_2)]$$

$$\Delta Y_1 = A_1[E_n(A_1+A_2) - E_n(A_1)]$$

Let us now introduce the probability of displacement U_A referred to all 2-calls. Then we obtain for ΔY_2 :

$$-\Delta Y_2 = U_A \cdot \lambda_2 \cdot h = U_A \cdot A_2$$

That means that from the λ_2 2-calls, arriving per time-unit, the U_A-th part is displaced. These displaced 2-calls would have had a further mean holding time h.

So the probability of displacement U_A is given by:

$$U_A = \frac{A_1}{A_2} \cdot [E_n(A_1+A_2) - E_n(A_1)] \qquad (15a)$$

It will be useful to define a further probability of displacement U_Y referred only to those 2-calls which could occupy a line.

$$U_Y = \frac{U_A}{1-E_n(A_1+A_2)} = \frac{A_1}{A_2} \cdot \frac{E_n(A_1+A_2) - E_n(A_1)}{1-E_n(A_1+A_2)} \qquad (15b)$$

These probabilities of displacement are shown in Fig. 1 and 2.

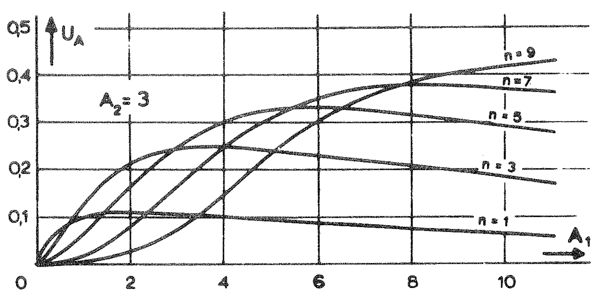

Fig. 1. The probabilities of displacement U_A as a function of the offered traffic A_1.

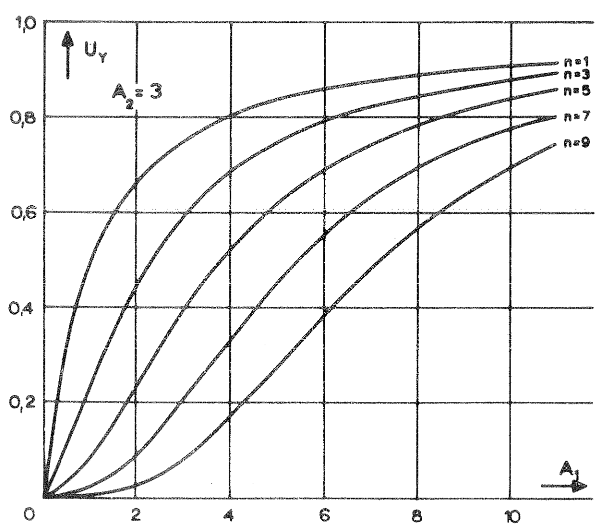

Fig. 2. The probability of displacement U_Y as a function of the offered traffic A_1.

On the other hand ΔY_2 can be expressed in the following manner:

$$-\Delta Y_2 = \lambda_2(1-B_2)h - \lambda_2(1-B_2)h_2 = A_2(1-B_2)(1-\frac{h_2}{h})$$

This enables us to calculate the mean holding time h_2

$$A_2[1-E_n(A_1+A_2)](1-\frac{h_2}{h}) = A_1[E_n(A_1+A_2) - E_n(A_1)]$$

$$h_2 = \{1- \frac{A_1[E_n(A_1+A_2) - E_n(A_1)]}{A_2[1-E_n(A_1+A_2)]}\}h = (1-U_Y)h \qquad (16)$$

From eq.(14,15a,16) we derive

$$Y_2 = A_2[1 - E_n(A_1+A_2)] - A_1[E_n(A_1+A_2) - E_n(A_1)]$$

$$= A_2(1-B_2-U_A) \qquad (14a)$$

3. The holding time distributions

3.1 Discipline: first-come, last-displaced

As the holding time distribution function (d.f.) of 1-calls is negative-exponential with the mean h, this d.f. needs not to be determined. For class 2, however, three d.f. are of interest: The holding time d.f. of all 2-calls, of the displaced 2-calls and finally of the nondisplaced 2-calls.

3.1.1 The difference-differential equation for the holding time d.f. of all 2-calls

As the holding time d.f. of the displaced or nondisplaced 2-calls, resp., can be derived from the d.f. of all 2-calls, it will be useful to determine the holding time d.f. for all 2-calls at first.

This can be done by means of a solution method, published among others by C.Palm /3/ :

A 2-call occupies a free line of a trunkgroup with $(x-1) < n$ busy lines. What will happen during the immediately following time interval dt ? Three events will influence the holding time of this considered call:

a) None of the existing (x-1) established calls ends and no 1-call arrives. (2-calls may arrive without consequence, they are, if at all, displaced before the considered 2-call, and therefore they are not relevant to its holding time). The state "x lines occupied" is not changed during dt.

b) No call of class 1 arrives, but one of the (x-1) established calls ends, which existed before the arrival of the considered call. (It is meaningless whether a 1- or a 2-call ends, as all existing 2-calls will be displaced, if at all, after the considered 2-call. Therefore it makes no difference for the considered 2-call, whether one of the (x-1) lines is occupied by a call of class 1 or 2). In this case the considered call continues its occupation in the new state (x-1).

c) A 1-call arrives and no call ends. Then the considered call continues its occupation in the new state (x+1).

With respect to the limitting-process $dt \to 0$ multiple events (arrivals and/or terminations) have probabilities of higher order in dt and need not be regarded.

The holding time d.f. of a 2-call, arriving during the state (x-1), will be denoted by $P_x(>t)$. Therefore $P_x(>t)$ denotes the probability that a 2-call, beginning its occupation in the state x, has a duration of greater than t.

The probability $P_x(>t+dt)$ is composed of the probabilities $P_x(>t)$, $P_{x+1}(>t)$ and $P_{x-1}(>t)$, multiplied by the corresponding probabilities of remaining in the state x or arriving at the states (x-1) or (x+1) during the time interval dt.

For $x < n$ holds:

$$P_x(>t+dt) = (1 - \lambda_1 dt - x\mu dt)\cdot P_x(>t) + \mu(x-1)dt\cdot P_{x-1}(>t) + \lambda_1 dt\cdot P_{x+1}(>t)$$

$$\frac{P_x(>t+dt) - P_x(>t)}{dt} = -(\lambda_1+\mu x)\cdot P_x(>t) + \mu(x-1)\cdot P_{x-1}(>t) + \lambda_1\cdot P_{x+1}(>t)$$

with $\tau = \frac{t}{h}$ we get:

$$\frac{dP_x(>\tau)}{d\tau} = -(A_1+x)\cdot P_x(>\tau) + (x-1)\cdot P_{x-1}(>\tau) + A_1\cdot P_{x+1}(>\tau) \qquad (17)$$

In the same way we obtain for $x = n$:

$$\frac{dP_n(>\tau)}{d\tau} = -(A_1+n)\cdot P_n(>\tau) + (n-1)\cdot P_{n-1}(>\tau) \qquad (17a)$$

The equations (17,17a) are a set of n differential equations. It can be solved with the initial conditions

$$P_x\ (>0) = 1 \quad \text{for } x = 1,2, \dots n. \qquad (18)$$

Equation (17) may be treated as one difference-differential equation with the boundary conditions

$$P_o(>\tau) = P_{n+1}(>\tau) = 0. \qquad (19)$$

3.1.2 The solution of the difference-differential equation

To solve this difference-differential equation we introduce the Laplace transform

$$g_x(s) = L\{P_x(>\tau)\} \qquad (20)$$

If we arrange eq. (17) with the aid of (19,20) in the usual matrix representation, we get:

$$\begin{array}{|cccccc|c}
g_1(s) & g_2(s) & g_3(s) & \dots & & g_n(s) & \\
\hline
A_1+1+s & -A_1 & & & & 0 & 1 \\
-1 & A_1+2+s & -A_1 & & & & 1 \\
 & -2 & A_1+3+s & -A_1 & & & 1 \\
 & & \ddots & \ddots & \ddots & & \vdots \\
 & 0 & & & & & \vdots \\
 & & & & -(n-1) & A_1+n+s & 1
\end{array} \qquad (21)$$

As s appears only in the leading diagonal of the matrix, it is obvious that for $g_x(s)$ holds:

$$g_x(s) = \frac{Z_x(s)}{D_n(s)} \qquad (22)$$

Where $Z_x(s)$ is a polynomial in s of degree n-1 and $D_n(s)$ is the solution of the determinant, which is a polynomial of the n-th degree.

At first a solution for $D_n(s)$ or its coefficients, resp., will be achieved. If we evaluate the determinant, beginning with the n-th row, we obtain:

$$D_n(s) = (A_1+n+s)D_{n-1}(s) - (n-1)A_1 D_{n-2}(s) \qquad (23)$$

where $D_{n-1}(s)$, $D_{n-2}(s)$ are the determinants of a system with (n-1) or (n-2) lines, resp.

As $D_j(s)$ (j= 1,2, ... n) is a polynomial of j-th degree, we write:

$$D_j(s) = \sum_{i=0}^{j} d_{i,j}\cdot s^i \qquad (24)$$

and introduce it into eq. (23). Thus we get a partial difference equation for $d_{i,j}$:

$$d_{i,j+2} - (A_1+2+j)\cdot d_{i,j+1} - d_{i-1,j+1} + (j+1)A_1\cdot d_{i,j} = 0 \qquad (25)$$

The boundary conditions can be taken from eq.(21)

Putting i = 0 in eq.(25) we get an ordinary difference equation, which has the following solution:

$$d_{o,j} = j!\sum_{\kappa=0}^{j} \frac{A_1^\kappa}{\kappa!} \qquad (26)$$

If we assume that in eq.(25) $d_{i-1,j+1}$ is a known function, we can derive a recurrence formula for $d_{i,j}$:

$$d_{i,j} = j!\sum_{\kappa=0}^{j} \frac{A_1^\kappa}{(\kappa+1)!}\cdot\sum_{\nu=i-1}^{\kappa} \frac{d_{i-1,\nu}}{A_1^\nu} \qquad (27)$$

So all coefficients $d_{i,j}$ (j= 1,2, ... n and i = 0,1,2, ... j) can be determined, if we begin with the determination of $d_{o,j}$, continue with $d_{1,j}$ and so on. Thus all coefficients $d_{i,n}$ (i = 0,1,2, ...n) of the determinant $D_n(s)$ are known.

In a similar way the polynomial $Z_x(s)$ will be obtained. At first we set

$$Z_x(s) = \sum_{i=0}^{n-1} c_{i,x}\cdot s^i \qquad (28)$$

By introducing this expression into eq.(22) and using eq. (21) we get:

$$-x\cdot c_{i,x} + (A_1+1+x)\cdot c_{i,x+1} + c_{i-1,x+1} - A_1\cdot c_{i,x+2} = d_{i,n} \qquad (29)$$

The boundary conditions are:

$$c_{i,o} = c_{i,n+1} = 0$$

$$c_{i,x} = 0 \quad \text{for } i<0 \text{ and } i>x.$$

Similarly to the solution of eq. (25) we obtain for $c_{o,x}$:

$$c_{o,x} = d_{o,n}\left[1 - \frac{E_n(A_1)}{E_{x-1}(A_1)}\right] \qquad (30)$$

where

$$E_j(A_1) = \frac{\frac{A_1^j}{j!}}{\sum_{i=0}^{j} \frac{A_1^i}{i!}}$$

For $c_{i,x}$ we find the following recurrence formula:

$$c_{i,x} = d_{i,n}\left[1 - \frac{E_n(A_1)}{E_{x-1}(A_1)}\right] + \frac{(x-1)!}{A_1^x}\sum_{\kappa=0}^{x-1}\frac{A_1^\kappa}{\kappa!}\sum_{\nu=1}^{\kappa} c_{i-1,\nu} - \frac{n!}{A_1^{n+1}}\cdot\frac{E_n(A_1)}{E_{x-1}(A_1)}\sum_{\kappa=0}^{n}\frac{A_1^\kappa}{\kappa!}\sum_{\nu=1}^{\kappa} c_{i-1,\nu} \qquad (31)$$

To determine the inverse Laplace transform of $g_x(s)$ it is necessary to find the roots of the denominator $D_n(s)$. As all coefficients of this polynomial $D_n(s)$ are known, the roots can be numerically computed. In /2/ a general proof, showing that all roots are simple, negative and real, is given. So we can decompose $g_x(s)$ into partial fractions:

$$g_x(s) = \frac{Z_x(s)}{D_n(s)} = \sum_{i=1}^{n} \frac{Z_x(s_i)}{\frac{d}{ds}\left[D_n(s)\right]\Big|_{s=s_i}} \cdot \frac{1}{s-s_i}$$

The inverse Laplace transform is given by

$$P_x(>\tau) = \sum_{i=1}^{n} \frac{\sum_{j=0}^{n-1} c_{j,x} \cdot s_i^j}{\sum_{j=1}^{n} j d_{j,n} \cdot s_i^{j-1}} \cdot \exp(s_i\tau) =$$

$$= \sum_{i=1}^{n} K_{i,x} \cdot \exp(s_i\tau) \qquad (32)$$

where

$$K_{i,x} = \frac{\sum_{j=0}^{n-1} c_{j,x} \cdot s_i^j}{\sum_{j=1}^{n} j d_{j,n} \cdot s_i^{j-1}} \qquad (33)$$

The d.f. $P_x(>\tau)$ is depicted in Fig. 3 for n= 5 lines. Note that 2-calls, starting in the state n, have a noticeably reduced mean holding time.

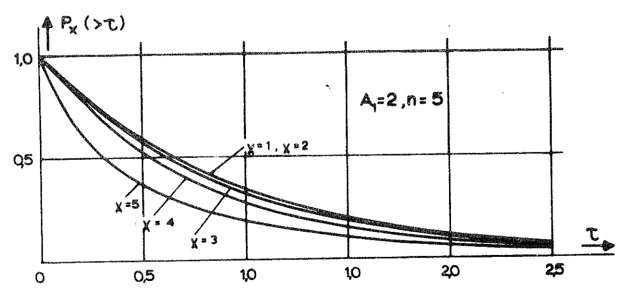

Fig. 3: The distribution function $P_x(>\tau)$.

3.1.3 The distribution function $P(>\tau)$ of an arbitrary , established 2-call

The probability $P(>\tau)$ of an arbitrary 2-call, not regarding the state x in which it starts, can be found by multiplying the probability $P_x(>\tau)$ by the probability p_x^* and summarizing these terms from x=1 to x=n.

The probability p_x^* denotes the probability that a 2-call starts in the state x. It can easily be derived from the state probabilities. It holds:

$$p_x^* = \frac{p_{x-1}}{1-E_n(A_1+A_2)} \qquad (34)$$

This means that a 2-call starts in the state x when it finds the state (x-1)<n at the time of its arrival.

The holding time d.f. of all established 2-calls therefore is given by

$$P(>\tau) = \sum_{x=1}^{n} p_x^* \cdot P_x(>\tau)$$

By using eq. (32) we obtain

$$P(>\tau) = \sum_{x=1}^{n} p_x^* \cdot \sum_{i=1}^{n} K_{i,x} \cdot \exp(s_i\tau)$$

$$= \sum_{i=1}^{n} C_i \cdot \exp(s_i\tau) \qquad (35)$$

where

$$C_i = \sum_{x=1}^{n} p_x^* \cdot K_{i,x} \qquad (35a)$$

As a next step we define a density function, which is normalized to the mean holding time h:

$$p(\tau) = -\frac{dP(>\tau)}{d\tau} = \sum_{i=1}^{n} (-s_i) C_i \cdot \exp(s_i\tau)$$

$$= \sum_{i=1}^{n} b_i \cdot \exp(s_i\tau) \qquad (36)$$

where

$$b_i = -s_i C_i \qquad (36a)$$

3.1.4 The normalized density functions $p_s(\tau)$ and $p_d(\tau)$ referred to the successful or displaced 2-calls, resp.

The density function $p(\tau)$ for all established 2-calls is composed of $p_s(\tau)$ and $p_d(\tau)$ in the following way:

$$(1-U_Y) \cdot p_s(\tau) + U_Y \cdot p_d(\tau) = p(\tau) \qquad (37)$$

On the other hand the density function of all established 2-calls including the displacing calls of class 1 is negative exponential. So we can write:

$$(1-U_Y) \cdot p_s(\tau) + U_Y \cdot p_{1+2}(\tau) = \exp(-\tau) \qquad (38)$$

The density function $p_{1+2}(\tau)$ can be obtained by folding the density function $p_d(\tau)$ of the displaced 2-calls with the density function $\exp(-\tau)$ of the displacing 1-calls. Thus we obtain:

$$p_{1+2}(\tau) = \int_{\zeta=0}^{\tau} p_d(\tau-\zeta) \exp(-\zeta) d\zeta \qquad (39)$$

By introducing eq. (39) in eq.(38) and subtracting eq. (37) from eq. (38) we get

$$\int_{\zeta=0}^{\tau} p_d(\tau-\zeta) \exp(-\zeta) d\zeta - p_d(\tau) = \frac{1}{U_Y}\left[\exp(-\tau) - p(\tau)\right] \qquad (40)$$

As it holds $s_i \neq -1$ for all i (cf. /2/), it can be shown that

$$p_d(\tau) = \sum_{i=1}^{n} \alpha_i \cdot \exp(s_i\tau) \qquad (41)$$

is the solution of eq. (40), where

$$\alpha_i = \frac{b_i}{U_Y} \cdot \frac{1+s_i}{s_i} \qquad i = 1,2, \ldots n \qquad (42)$$

As U_Y, b_i, and s_i are given (eq. (15a,36a)),the desired density function $p_d(\tau)$ is determined.

In a similar way we get

$$p_s(\tau) = \sum_{i=1}^{n} \beta_i \cdot \exp(s_i\tau) \qquad (43)$$

where

$$\beta_i = -\frac{1}{1-U_Y} \cdot \frac{b_i}{s_i} \qquad (44)$$

These results enable us to determine the distribution functions $P_d(>\tau)$ and $P_s(>\tau)$. We find by integration

$$P_d(>\tau) = -\frac{1}{U_Y}\sum_{i=1}^{n}\frac{1+s_i}{s_i^2}b_i\cdot\exp(s_i\tau) \quad (45)$$

$$P_s(>\tau) = +\frac{1}{1-U_Y}\sum_{i=1}^{n}\frac{1}{s_i^2}b_i\cdot\exp(s_i\tau) \quad (46)$$

From these d.f. we can derive the mean holding times:

$$\bar{\tau}_2 = \sum_{i=1}^{n}\frac{1}{s_i^2}b_i \qquad h_2 = \bar{\tau}_2\cdot h \quad (47)$$

$$\bar{\tau}_{2,d} = \frac{1}{U_Y}\sum_{i=1}^{n}\frac{1+s_i}{s_i^3}b_i \quad h_{2,d} = \bar{\tau}_2\cdot h \quad (48)$$

$$\bar{\tau}_{2,s} = -\frac{1}{1-U_Y}\sum_{i=1}^{n}\frac{1}{s_i^3}b_i \quad h_{2,s} = \bar{\tau}_2\cdot h \quad (49)$$

In Fig. 4 the d.f. $P(>\tau)$, $P_s(>\tau)$ and $P_d(>\tau)$ are shown. In addition, the function $\exp(-\tau)$ is depicted as a comparison. Fig. 5 represents the mean holding times as a function of the offered traffic A_1.

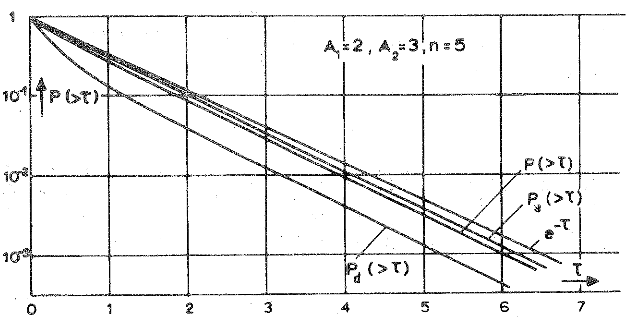

Fig. 4: The d.f. $P(>\tau)$, $P_s(>\tau)$ and $P_d(>\tau)$

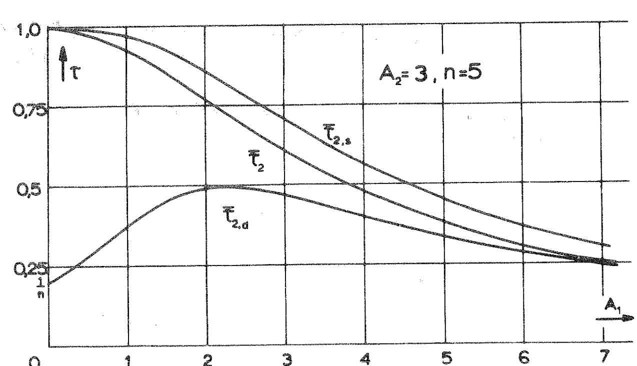

Fig. 5: The mean holding times $\bar{\tau}_2$, $\bar{\tau}_{2,d}$, $\bar{\tau}_{2,s}$.

3.2 Discipline: first-come, first-displaced

This discipline will be especially useful, when the importance of a call decreases with the elapsing time. Then it is appropriate that a call, which finds all lines busy, is allowed to displace a call of its own class, too. Typical applications of this discipline are possible in the field of flight-traffic control and weather-service.

If we allow displacements within the own class, the formulas for loss and the probability of displacement according to eq.(10,11,13,15) will hold no longer. The probabilities of state, however, are not influenced anyhow. This enables us to consider two classes only as in chapter 3.1.

3.2.1 The probability of loss and the probability of displacement, the carried traffic, mean holding time

Class 1

A 1-call will never be rejected

$$B_1 = 0 \quad (50)$$

A 1-call will be displaced, if an arriving 1-call finds all lines occupied by calls of class 1 only.

$$U_{A_1} = U_{Y_1} = E_n(A_1) \quad (51)$$

The carried traffic is given by

$$Y_1 = \lambda_1\cdot h - \lambda_1\cdot U_{Y_1}\cdot h = A_1[1 - E_n(A_1)] \quad (52)$$

and the mean holding time for all 1-calls is according to eq. (11) :

$$h_1 = \frac{Y_1\cdot h}{(1-B_1)A_1} = [1-E_n(A_1)]h = (1-U_{Y_1})h \quad (53)$$

Class 2

A 2-call is rejected only, if an arriving 2-call finds all lines occupied by 1-calls only

$$B_2 = E_n(A_1) \quad (54)$$

The probability of displacement is composed of two parts: 2-calls may either be displaced by 1-calls or by calls of their own class. Therefore it holds

$$U_{A_2} = \frac{A_1}{A_2}[E_n(A_1+A_2) - E_n(A_1)] + [E_n(A_1+A_2) - E_n(A_1)]$$

$$= \frac{A_1+A_2}{A_2}[E_n(A_1+A_2) - E_n(A_1)] \quad (55)$$

$$U_{Y_2} = \frac{U_{A_2}}{1-E_n(A_1)} = \frac{A_1+A_2}{A_2[1-E_n(A_1)]}[E_n(A_1+A_2)-E_n(A_1)] \quad (56)$$

Thereby we obtain the carried traffic Y_2:

$$Y_2 = \lambda_2(1-B_2)h - \lambda_2 U_{A_2} h$$

$$= A_2[1-E_n(A_1)] - (A_1+A_2)[E_n(A_1+A_2) - E_n(A_1)]$$

$$= A_2[1-E_n(A_1+A_2)] - A_1[E_n(A_1+A_2) - E_n(A_1)] \quad (58)$$

This result is identical with that of eq. (14a). This is evident, because displacements within the same class have no influence on the carried traffic of this class.

The mean holding time of all 2-calls is given by

$$h_2 = \frac{Y_2 h}{(1-B_2)A_2} = \{1 - \frac{(A_1+A_2)[E_n(A_1+A_2)-E_n(A_1)]}{A_2[1 - E_n(A_1)]}\}h$$

$$= (1 - U_{Y_2})h \quad (59)$$

3.2.2 The holding time distributions

It is not necessary to derive these distribution functions separately for class 1 and for class 2. The results to be derived for class 2 will hold also for class 1, if we replace A_2 by A_1 and A_1 by $A_0 = 0$. (In this way eq. (50,51,52,53) can be derived from eq. (54,55,56,58,59), too).

Similarly as shown in chapter 3.1 we obtain the corresponding difference-differential equation:

$x = 1,2, \dots n$

$$\frac{d}{d\tau}P_x(>\tau) = -(x+A_1+A_2)\cdot P_x(>\tau) + (x-1)\cdot P_{x-1}(>\tau) + (A_1+A_2)\cdot P_{x+1}(>\tau) \quad (60)$$

The boundary conditions with respect to x are given by

$$P_o(>\tau) = P_{n+1}(>\tau) = 0 \quad (61)$$

and the initial conditions with respect to τ

$$P_x(>0) = 1 \quad x = 1,2, \dots n$$

It is not necessary to develop a solution for eq. (60). This equation is identical to eq.(17), if we replace A_1 in (17) by A_1+A_2. Therefore we can calculate the holding time distribution $P_x(>\tau)$ for this discipline in the same way as we did for the discipline "first-come, last-displaced".

The analogous problem without displacements by calls of the own class can be solved, too. In this case eq.(60) holds only for $x = 1,2, \dots n-1$; for $x = n$ we obtain:

$$\frac{d}{d\tau}P_n(>\tau) = -(n+A_1)\cdot P_n(>\tau) + (n-1)\cdot P_{n-1}(>\tau) \quad (60a)$$

A 2-call, arriving when all lines are busy, is then lost and cannot influence the state "n lines occupied".

But now equation (60) loses its regularity and cannot be solved as described in chapter 3.1. An analogous solution method as shown in chapter 3.3 will, however, be possible.

To derive the d.f. for all calls of class 2, we need the probability that a 2-call starts in the state x. According to the discipline "first-come, first-displaced" only existing 1-calls are relevant to an arriving 2-call. The reason for this is that an established 2-call is, if at all, always displaced before the considered 2-call. Conversely, a 2-call, which arrives after the considered 2-call, will affect the holding time of the considered call. This explains, why now in eq.(60) the term A_1+A_2 appears instead of A_1 in eq.(17). Therefore we obtain:

$$p_x^* = \frac{p_{x_1-1}}{1-E_n(A_1)} \quad (64)$$

That means, we can calculate p_x^* in the same manner as in chapter 3.1, but now we have to replace A_1+A_2 in eq.(34) by A_1.

3.2.3 The distribution functions $P(>\tau)$, $P_s(>\tau)$ and $P_d(>\tau)$

To determine the d.f. $P(>\tau)$, $P_s(>\tau)$ and $P_d(>\tau)$ as well as the corresponding mean holding times, we can proceed in the same way as in chapter 3.1.

To solve the problem for r>2 priority classes, the rules, presented in chapter 2, are valid.

3.3 The discipline: random-displacement

3.3.1 The probabilities of loss and displacement, mean holding time

Henceforth we don't admit displacements within the own class. The results for the probabilities of loss or displacement, resp., the carried traffic and the mean holding time of all 1-calls or of all 2-calls can be taken therefore from chapter 2.

3.3.2 The holding time distributions

Up to now only the quantity of established calls of class 1 and 2 (cf. 3.1) or only of class 1 (cf. 3.2) existing in the system were significant for an arriving 2-call. Now it is necessary to know the number x_1 of existing 1-calls and the number x_2 of existing 2-calls, separately. The reason is that in the state "n lines busy" the probability w of an established 2-call to be selected for displacement depends on the number x_2 of existing 2-calls. It holds:

$$w = \frac{x_2-1}{x_2}$$

Thus it is inevitable to set up a partial difference-differential equation for the d.f. $P_{x_1,x_2}(>\tau)$ of the 2-calls. We obtain analogously to chapter 3.1

for $x_1 = 0,1, \dots n-1$; $x_2 = 1,2, \dots n-1$
$x_1+x_2 = 1,2, \dots n-1$

$$\begin{aligned}\frac{d}{d\tau}P_{x_1,x_2}(>\tau) = &-(x_1+x_2+A_1+A_2)\cdot P_{x_1,x_2}(>\tau) \\ &+A_1\cdot P_{x_1+1,x_2}(>\tau) + A_2\cdot P_{x_1,x_2+1}(>\tau) \\ &+x_1\cdot P_{x_1-1,x_2}(>\tau) \\ &+(x_2-1)\cdot P_{x_1,x_2-1}(>\tau)\end{aligned} \quad (65)$$

and for $x_1 = 0,1, \dots n-1$; $x_2 = 1,2, \dots n$
$x_1+x_2 = n$

$$\begin{aligned}\frac{d}{d\tau}P_{x_1,x_2}(>\tau) = &-(x_1+x_2+A_1)\cdot P_{x_1,x_2}(>\tau) \\ &+x_1\cdot P_{x_1-1,x_2}(>\tau) \\ &+(x_2-1)\cdot P_{x_1,x_2-1}(>\tau) \\ &+A_1\frac{x_2-1}{x_2}\cdot P_{x_1+1,x_2-1}(>\tau)\end{aligned} \quad (66)$$

The boundary conditions are given by

$$P_{x_1,x_2}(>\tau) = 0 \quad x_1 = 0,1, \dots n;\ x_2 = 0 \quad (67)$$

and the initial conditions

$$P_{x_1,x_2}(>0) = 1 \quad x_1 = 0,1, \dots n-1;\ x_2 = 1,2, \dots n \quad (68)$$

A handy general expression for $P_{x_1,x_2}(>\tau)$ with the aid of eq. (65) cannot be derived, because eq. (65) represents a partial difference-differential equation with nonconstant coefficients with respect to the variables x_1, x_2. Moreover, this equation is additionally complicated by the boundary equation (66).

Besides a direct solution of the set of differential equations (65,66), which is, of course, only reasonable for small systems of equations, a numerical computation of the distribution functions $P_{x_1,x_2}(>\tau)$ is possible as shown below:

We define a Laplace transform

$$g_{x_1,x_2}(s) = L\{P_{x_1,x_2}(>\tau)\} = \frac{Z_{x_1,x_2}(s)}{D_m(s)}$$

and introduce it in the eq. (65,66). If we write down these equations for $g_{x_1,x_2}(s)$ in the matrix notation, we get a matrix of the rank $m = (n+1)n/2$.

The polynomials $D_m(s)$ can be developed by inserting m+1 appropriate values for s (e.g. s = =0,1,2, ...) in the determinant. Thus we obtain m+1 pairs of values which consist of values for s and the resulting values of the determinant.

By application of these pairs of values the coefficients of the polynomial $D_m(s)$ can be calculated by solving a set of m+1 linear equations.

In the same way the m polynomials $Z_{x_1,x_2}(s)$ can be found by inserting m values for each polynomial $Z_{x_1,x_2}(s)$ in those determinants which are obtained by substituting the "right side" of the matrix for the corresponding column of the determinant. All these polynomials $Z_{x_1,x_2}(s)$ are of degree (m-1).

By decomposing $g_{x_1,x_2}(s)$ into partial fractions and applying the inverse Laplace transform we obtain finally $P_{x_1,x_2}(>\tau)$.

To get the desired d.f. $P(>\tau)$, we need in addition to $P_{x_1,x_2}(>\tau)$ the probability of state p_{x_1,x_2}. For these probabilities holds:

$x_1 = 0,1,2, \ldots n-1;\ x_2 = 0,1,2, \ldots n-1;$
$x_1+x_2 = 0,1,2, \ldots n-1$

$$(A_1+A_2+x_1+x_2)\cdot p_{x_1,x_2} = A_1\cdot p_{x_1-1,x_2} + A_2\cdot p_{x_1,x_2-1} + (x_1+1)\cdot p_{x_1+1,x_2} + (x_2+1)\cdot p_{x_1,x_2+1} \qquad (69)$$

$x_1+x_2 = n;\ x_2 = 1,2, \ldots n$

$$(A_1+x_1+x_2)\cdot p_{x_1,x_2} = A_1\cdot p_{x_1-1,x_2} + A_2\cdot p_{x_1,x_2-1} + A_1\cdot p_{x_1-1,x_2+1} \qquad (70)$$

Together with the normalizing condition

$$\sum_{x_1=0}^{n} \sum_{x_2=0}^{n-x_1} p_{x_1,x_2} = 1 \qquad (71)$$

a set of m = (n+2)(n+1)/2 equations (69,70,71) is given to determine the m state probabilities. The solution of this set of equations can efficiently be done by iteration methods, e.g. by the Gauss-Seidel method, which is especially suitable for solutions on a computer. This method has the advantage of a small memory-requirement and provides a high accuracy of the results.

With the solution of the state probabilities p_{x_1,x_2} we can find analogously to eq. (34) the probability that a 2-call starts in the state (x_1,x_2).

3.3.3 The distribution functions $P(>\tau)$, $P_d(>\tau)$ and $P_s(>\tau)$, mean holding times

The d.f. of all 2-calls as well as those of the displaced or succesful, nondisplaced 2-calls, resp., and the mean holding times can be calculated in the same way as shown in chapter 3.1.

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