GRADED DELAY SYSTEMS WITH INFINITE OR FINITE SOURCE TRAFFIC AND EXPONENTIAL OR CONSTANT HOLDING TIME

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ABSTRACT

The paper deals with single-stage delay systems of the types M/M/n and M/D/n and single-stage delay-loss systems of the type M/M/n-s both in case of full or limited accessibility. Further distinctions are made between single and multi-queues, ideal and real gradings, and a finite or infinite number of traffic sources. Waiting calls are served acc. to the disciplines FIFO, RANDOM, or LIFO.

Starting from known results, various systems are reviewed systematically with respect to their stationary state probabilities, characteristic mean values and waiting time distributions. Curves of calculated results are given and compared with simulations.

1. INTRODUCTION

1.1 PROBLEM

In certain types of telephone and data switching systems the data traffic flow from peripheral to centralized devices is handled by concentrating switching networks. Usually, these access networks are operated as delay systems. To design such networks, reliable methods for grade-of-service calculation are necessary.
To describe the traffic flow in such systems, an adequate queuing model is constructed consisting of servers and waiting places for queuing. Arriving requests (calls) are generated by a finite or an infinite number of traffic sources. If there are multiple servers a certain server is selected acc. to a hunting discipline. If all accessible servers are occupied, an arriving call may wait if there is a waiting place available. Waiting calls are selected for service acc. to a certain interqueue or queue discipline.

Input and service processses of the queuing model are described by distribution functions (df) of interarrival times of calls and service times, respectively.

The queuing model is analyzed by considering the discrete stochastic process of the random number of calls within the system. Under the assumption of stationarity the steady-state probabilites are obtained from which other characteristic values are derived for system dimensioning as e.g. mean waiting times.

In the past a number of solutions has been derived for single-stage delay systems. These solutions will now be extended with respect to criteria as finite sources, real gradings or limited waiting capacity, respectively. The results are part of a table book on delay systems |20| and are also used as basic modules for the analysis of link systems with waiting |13|.

1.2 SURVEY OF ANALYZED QUEUING MODELS

To represent the queuing models briefly, Kendall's short notation will be used with some modifications as follows:

X/Y/n(k)-s

where
- X type of arrival process, e.g. M: Markovian
- Y type of service process, e.g. D: Deterministic
- n number of servers
- k accessibility (if limited)
- s number of waiting places (if limited)

With this notation an overview on the problems dealt in this paper is given in Table 1.

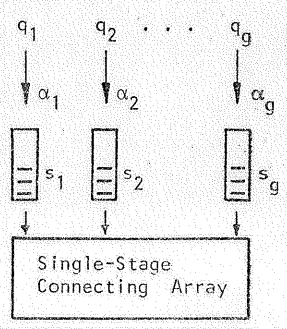
TYPE OF QUEUING MODEL		DELAY SYSTEMS M/M/n(k) $q \to \infty$	DELAY SYSTEMS M/M/n(k) $q < \infty$	DELAY SYSTEMS M/D/n(k) $q \to \infty$	DELAY-LOSS SYSTEMS M/M/n(k)-s $q \to \infty$	DELAY-LOSS SYSTEMS M/M/n(k)-s $q < \infty$
		Chapter 3		Chapter 4	Chapter 5	
Full Access	Single Queue	3.1	3.2	4.1	5.1	5.2
Full Access	Multi-Queue	3.3	3.4	4.1	5.3	5.4
Limited Access	Ideal Grading	3.5	3.6	4.2	5.5	5.6
Limited Access	Real Grading	3.5	3.6	4.2	5.5	5.6

Table 1: Survey of queuing models.
(q = number of traffic sources)

2. GENERAL QUEUING MODEL

2.1 SYSTEM STRUCTURE AND OPERATING MODES

The system structure of the general queuing model is shown in Fig.1.



- g number of groups (classes)
- q_j number of sources of group j
- α_j idle source arrival rate in group j
- s_j number of waiting places of queue j
- n number of servers
- k accessibility
- h mean service time

Fig.1: General queuing model.

Arriving calls hunt the servers with full or limited accessibility, cf. Fig.2.

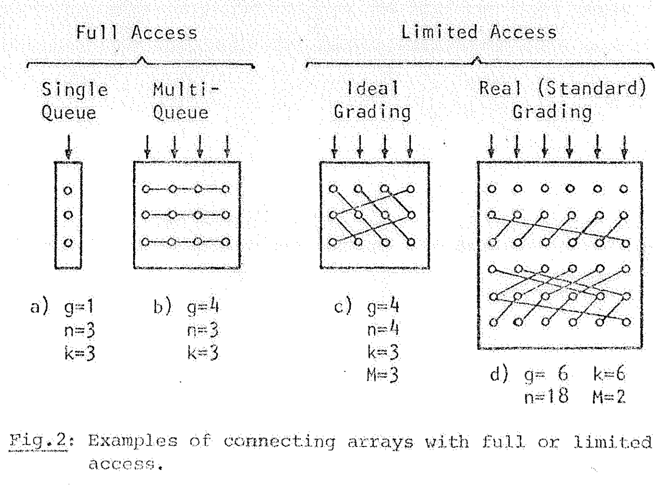


Fig.2: Examples of connecting arrays with full or limited access.

The special interconnection scheme in case of limited acces groups is called "grading". Two main types will be distinguished:

- "ideal" gradings with a combinatorial number of grading groups $g = \binom{n}{k}$ in case of random hunting or $g = \binom{n}{k}k!$ in case of sequential hunting, and
- nonideal ("real") gradings.

In the latter case only standard gradings |11| will be considered. From n,k and g the mean interconnecting number is defined as $M = g \cdot k / n$.

Arriving calls are handled acc. to certain operating strategies comprising hunting mode, queue and interqueue disciplines. As hunting mode we assume random or sequential hunting. For nonideal gradings only sequential hunting will be regarded. If no idle server is accessible, the incoming call occupies a waiting place in its group; if all available waiting places are occupied, the incoming call is lost.

Waiting calls are selected for service acc. to operating modes as interqueue disciplines (selection of a certain queue in a multi-queue system), queue disciplines (selection of a call within a queue), or disciplines with regard to all waiting calls in a multi-queue system. In this paper the basic disciplines FIFO (first-in, first-out), RANDOM, and LIFO (last-in, first-out) will be considered. When a call is selected for service, its waiting place is released immediately.

2.2 ARRIVAL AND SERVICE PROCESSES

Arriving calls are generated either by a finite number q of traffic sources with an idle source arrival rate α or by an infinite number $q \to \infty$ of traffic sources with a constant arrival rate λ. Throughout the paper only negative exponential df are assumed for the random interarrival times T_A, i.e.

$$A(t) = P\{T_A \le t\} = \begin{cases} 1 - e^{-\lambda t} & , \quad q \to \infty \\ 1 - e^{-(q-r)\alpha t} & , \quad q < \infty \end{cases} \tag{1}$$

where r the number of nonidle sources. The random service (holding) times T_H are mutual independent and identically distributed acc. to a negative exponential or constant df with mean h, i.e.

$$H(t) = P\{T_H \le t\} = \begin{cases} 1 - e^{-\varepsilon t} , & \text{Markovian (M)} \\ 0, \ t < h \\ 1, \ t \ge h \end{cases} , \quad \text{Deterministic (D)} \tag{2}$$

where $\varepsilon = 1/h$ the termination rate. The service time is independent of the source group by which a call has been generated. From λ, α, and h the offered traffic $A = \lambda \cdot h$ and the idle source offered traffic $\beta = \alpha \cdot h$ are defined in case of infinite or finite number of sources, respectively.

2.3 CHARACTERISTIC VALUES

Characteristic values are measures for the system performance (grade-of-service). They are derived from random variables describing the stochastic behaviour of the service system as, e.g., X (random number of calls in service), Z (random number of calls waiting), or T_W (random waiting time).

p(..)	stationary state probabilities seen from an outside observer
π(..)	stationary state probabilities seen from an arriving call
Y	carried traffic
Ω	mean queue length of waiting calls
W	probability of delay
B	probability of loss
w	mean waiting time of offered calls
t_W	mean waiting time of waiting calls
W(>t)	complementary df of waiting time

3. DELAY SYSTEMS WITH EXPONENTIAL HOLDING TIMES

In this chapter pure delay systems will be considered having single or many queues, infinite or finite number of traffic sources, and full or limited accessibility, respectively.

3.1 FULL ACCESS, INFINITE SOURCE, SINGLE QUEUE MODEL M/M/n

3.1.1 Stationary State Probabilities

To describe the system state, a single state variable i = 0,1,... is sufficient indicating the number of calls in the system. For reasons of conformity in this paper a distinction is made between the numbers x and z of calls being served or waiting by introduction of a pseudo-twodimensional state variable (x,z), cf. Fig.3.

$$\cdots \rightleftarrows (x-1,0) \underset{b(x,0)}{\overset{a(x-1,0)}{\rightleftarrows}} (x,0) \rightleftarrows \cdots \rightleftarrows (n,0) \rightleftarrows \cdots \rightleftarrows (n,z-1) \underset{b(n,z)}{\overset{a(n,z-1)}{\rightleftarrows}} (n,z) \rightleftarrows \cdots$$

Fig.3: State space and transitions of the M/M/n delay system with full access and a single queue.

In case of an infinite source traffic, the transition rates are

$$\begin{aligned} a(x,0) &= \lambda \quad , \quad x = 0,1,\dots,n-1 \\ a(n,z) &= \lambda \quad , \quad z = 0,1,\dots \end{aligned} \tag{3a}$$

$$\begin{aligned} b(x,0) &= x\varepsilon \quad , \quad x = 1,2,\dots,n-1 \\ b(n,z) &= n\varepsilon \quad , \quad z = 0,1,\dots \end{aligned} \tag{3b}$$

In the stationary state the equilibrium equations for the probabilities of state p(x,z) are:

$$p(x,0) \cdot b(x,0) = p(x-1,0) \cdot a(x-1,0) \quad , \quad x = 1,2,\dots,n, \tag{4a}$$

$$p(n,z) \cdot b(n,z) = p(n,z-1) \cdot a(n,z-1) \quad , \quad z = 1,2,\dots \tag{4b}$$

Introducing eq.(3a,b) into eq.(4a,b) we have

$$p(x,0) = p_0 \cdot \frac{A^x}{x!} \quad , \quad x \le n \tag{5a}$$

$$p(n,z) = p_0 \cdot \frac{A^n}{n!} \cdot \rho^z \quad , \quad z \ge 0, \tag{5b}$$

where $p_0 = p(0,0)$ and $\rho = A/n$.

The probability p_0 indicates generally that the system is in the idle state; it is found from the normalizing condition that the sum of all state probabilities must equate to 1.

3.1.2 Characteristic Values

The characteristic values are defined as follows:

$$Y = E[X] = \sum_{x=0}^{n-1} x \cdot p(x,0) + n \cdot \sum_{z=0}^{\infty} p(n,z) = A \tag{6}$$

$$\Omega = E[Z] = \sum_{z=0}^{\infty} z \cdot p(n,z) = p(n,0) \cdot \frac{\rho}{(1-\rho)^2} \tag{7}$$

$$W = P\{T_W > 0\} = \sum_{z=0}^{\infty} p(n,z) = p(n,0) \cdot \frac{1}{1-\rho} \tag{8}$$

$$w = E[T_W] = \Omega/\lambda = p(n,0) \cdot \frac{h}{n(1-\rho)^2} \tag{9a}$$

$$t_W = E[T_W | T_W > 0] = \frac{\Omega}{\lambda W} = \frac{h}{n(1-\rho)} . \tag{9b}$$

The df of waiting times depends on the underlying queue discipline. Generally it holds:

$$W(>t) = P\{T_W > t\} = \sum_{z=0}^{\infty} p(n,z) \cdot w(t|z) , \tag{10}$$

where w(t|z) the conditional complementary df of waiting times conditioned on the number z of waiting calls met by an arriving call. In case of FIFO

$$w(t|z) = P\{T_W > t | z\} = \sum_{i=0}^{z} \frac{(n\varepsilon t)^i}{i!} \cdot e^{-n\varepsilon t} \quad , \quad z = 0,1,\dots \tag{11a}$$

which results to

$$\frac{W(>t)}{W} = e^{-t/t_W} \tag{11b}$$

So far, solutions eq.(5) - (11) go back to A.K.Erlang |5|.

For the RANDOM queue discipline, the values $w(t|z)$ are solutions of a differential equation system formulated by C.Palm and E.Vaulot |7,25,28| for the random process of waiting (Kolmogorov-backward equations) :

$$w'(t|z) = -(\lambda+n\varepsilon)\cdot w(t|z) + \lambda w(t|z+1) + n\varepsilon\cdot\frac{z}{z+1}\cdot w(t|z-1)\ ,\ z\geq 0. \tag{12}$$

To evaluate the df of waiting times numerically, approximations are important based on the lower moments, viz.

$$E[T_W|T_W>0] = h/n(1-\varrho) \qquad = t_W, \tag{13a}$$
$$E[T_W^2|T_W>0] = 2h^2/n^2(1-\varrho)^2(1-\varrho/2) \tag{13b}$$
$$E[T_W^3|T_W>0] = 6h^3(4+2\varrho)/n^3(1-\varrho)^3(2-\varrho)^2 \tag{13c}$$

respectively the variation coefficient c_W acc. to

$$c_W^2 = \frac{E[T_W^2|T_W>0]}{E[T_W|T_W>0]^2} - 1 = \frac{2-\varrho}{2+\varrho}. \tag{13d}$$

J.Riordan |25| approximates $W(>t)$ by a hyperexponential df

$$\frac{W(>t)}{W} \cong p\cdot e^{-a_1 t} + (1-p)\cdot e^{-a_2 t} \tag{14}$$

where the parameters p, a_1, a_2 are determined from the first, second, and third moment acc.to eq.(13a-c). A new approximation |20,21| is based on the Weibull df:

$$\frac{W(>t)}{W} \cong e^{-(at)^b} \tag{15}$$

where the parameters a and b are determined only from the first and second moment by iteration acc. to

$$t_W = \Gamma(1+\tfrac{1}{b})/a \tag{16a}$$
$$c_W^2 = \frac{\Gamma(1+\frac{2}{b})}{\Gamma(1+\frac{1}{b})^2} - 1. \tag{16b}$$

This approximation fits better with simulation than eq.(14) although it is based on two moments only!

3.2 FULL ACCESS, FINITE SOURCE, SINGLE QUEUE MODEL M/M/n

3.2.1 Stationary State Probabilites

In a finite source pure delay system with a total number of q sources there are s = q-n waiting places available. The state space is similar to Fig.3 but limited by $z \leq z_{max} = q-n$. The transition rates are:

$$\begin{aligned} a(x,0) &= (q-x)\alpha &&, x = 0,1,\dots,n-1 \\ a(n,z) &= (q-n-z)\alpha &&, z = 0,1,\dots,q-n-1 \end{aligned} \tag{17a}$$

$$\begin{aligned} b(x,0) &= x\varepsilon &&, x = 1,2,\dots,n-1 \\ b(n,z) &= n\varepsilon &&, z = 0,1,\dots,q-n. \end{aligned} \tag{17b}$$

In the stationary state the equilibrium equations for the state probabilities $p(x,z)$ seen from an outside observer are given by eq.(4a,b) with finite state space (z_{max}=q-n). Introducing eq.(17a,b) into eq.(4a,b) we find

$$p(x,0) = p_0\cdot\frac{\beta^x}{x!}\cdot\prod_{i=0}^{x-1}(q-i) \qquad , x\leq n \tag{18a}$$

$$p(n,z) = p_0\cdot\frac{\beta^n}{n!}\cdot\left(\frac{\beta}{n}\right)^z\cdot\prod_{i=0}^{n+z-1}(q-i) \qquad , z\geq 0. \tag{18b}$$

An arriving call meets a state (x,z) with probability

$$\pi(x,z) = \frac{(q-x-z)\cdot p(x,z)}{\sum_{i=0}^{n-1}(q-i)p(i,0) + \sum_{j=0}^{q-n}(q-n-j)\cdot p(n,j)}, \quad \begin{aligned} x &= 0,1,\dots,n \\ z &= 0,1,\dots,q-n-1 \end{aligned} \tag{19}$$

3.2.2 Characteristic Values

Analogously to Section 3.1.2 holds

$$Y = A = \sum_{x=0}^{n-1} x\cdot p(x,0) + n\sum_{z=0}^{q-n} p(n,z) \tag{20}$$

$$\Omega = \sum_{z=0}^{q-n} z\cdot p(n,z) = q - A - \frac{A}{\beta} \tag{21}$$

$$W = \sum_{z=0}^{q-n-1} \pi(n,z) \tag{22}$$

$$w = \Omega/\lambda\ , \qquad t_W = w/W. \tag{23a,b}$$

The df of waiting times is determined by

$$W(>t) = \sum_{z=0}^{q-n-1} \pi(n,z)\cdot w(t|z). \tag{24}$$

In case of FIFO $w(t|z)$ is given by eq.(11a). Solutions eq.(18) - (24) were derived by F.L.Bauer and H.Störmer |3|. An approximate solution was reported by A.Lotze |22|.

For RANDOM service of waiting calls a system of differential equations has to be solved analogously to eq.(12). The equations read in the general case of z:

$$\begin{aligned} w'(t|z) = &-[(q-n-z-1)\alpha+n\varepsilon]\cdot w(t|z) + (q-n-z-1)\alpha\cdot w(t|z+1) \\ &+ n\varepsilon\cdot\frac{z}{z+1}\cdot w(t|z-1). \end{aligned} \tag{25}$$

The solution of eq.(25) incorporates the roots of a characteristic equation |26|. For numerical purposes we use the concept of approximating the conditional df $w(t|z)$ by exponential or hyperexponential df meeting the first, second and third moment of $w(t|z)$ exactly. This concept was introduced in case of systems M/M/n-s with RANDOM queue discipline and yields extremly high accuracy |18|. By this

$$w(t|z) \cong p_z\cdot e^{-a_{1z}t} + (1-p_z)\cdot e^{-a_{2z}t} \tag{26}$$

where the parameters p_z, a_{1z}, a_{2z} are determined from the first, second, and third moment of $w(t|z)$. These moments are obtained from linear systems of equations derived from eq.(25), |18,26| .

3.3 FULL ACCESS, INFINITE SOURCE, MULTI-QUEUE MODEL M/M/n

In this case the general delay system of Fig.1 is considered with $g>1$ input groups or classes of calls with arrival rates λ_j , j = 1,2,...,g .

3.3.1 Stationary State Probabilities

For a complete description of the system performance, various system state descriptions are defined which are also used for delay-loss systems (cf. Chapter 5) :

a) $(x_1,\dots,x_g;\ c_1,\dots,c_z)$ — x_i number of group-i-calls in service; c_j group index of that call waiting at the j-th position

b) $(x_1,\dots,x_g;\ z_1,\dots,z_g)$ — x_i as a); z_j number of group-j-calls waiting

c) $(x;\ c_1,\dots,c_z)$ — x total number of calls in service without regard to their origin; c_j as a)

d) $(x;\ z_1,\dots,z_g)$ — x as c); z_j as b)

e) $(x;\ z)$ — x as c); z total number of waiting calls

Because of full accessibility, the quantities c_j, z_j, and z are zero in all cases where $x = x_1+\dots+x_g < n$.

For any of the disciplines FIFO, RANDOM, or LIFO with respect to all waiting calls holds :

$$\text{a)} \quad p(x_1,\dots,x_g;c_1,\dots,c_z) = p_0 \cdot \prod_{i=1}^{g}\left(\frac{A_i^{x_i}}{x_i!}\right) \cdot \prod_{j=1}^{z} \varrho_{c_j} \tag{27}$$

$$\text{b)} \quad p(x_1,\dots,x_g;z_1,\dots,z_g) = p_0 \cdot \prod_{i=1}^{g}\left(\frac{A_i^{x_i}}{x_i!}\right) \cdot z! \cdot \prod_{j=1}^{g}\left(\frac{\varrho_j^{z_j}}{z_j!}\right) \tag{28}$$

$$\text{c)} \quad p(x;c_1,\dots,c_z) = \begin{cases} p_0 \cdot \dfrac{A^x}{x!} & , \; x < n \\ p_0 \cdot \dfrac{A^n}{n!} \cdot \prod\limits_{j=1}^{z} \varrho_{c_j} & , \; x = n \end{cases} \tag{29}$$

$$\text{d)} \quad p(x;z_1,\dots,z_g) = \begin{cases} p_0 \cdot \dfrac{A^x}{x!} & , \; x < n \\ p_0 \cdot \dfrac{A^n}{n!} \cdot \prod\limits_{j=1}^{g}\left(\dfrac{\varrho_j^{z_j}}{z_j!}\right) & , \; x = n \end{cases} \tag{30}$$

$$\text{e)} \quad p(x;z) = \begin{cases} p_0 \cdot \dfrac{A^x}{x!} & , \; x < n \\ p_0 \cdot \dfrac{A^n}{n!} \cdot \varrho^z & , \; x = n \end{cases} \tag{31}$$

where $A_i = \lambda_i / \varepsilon$, $A = A_1 + \dots + A_g$, $\rho_i = A_i / n$, $\rho = A/n$.

The proof of eq.(27) is carried out by insertion into the equilibrium state equation and stating consistency for all equations regarding the underlying queue discipline. Eq.(28) - (31) can be found either directly by the same technique or by combining equally probable microstates to macrostates. Eq.(27) has been proved in case of FIFO along with queuing network theorems |1|. In case of RANDOM eq.(30) was derived directly |15|.

Additionally, we state that there are g+1 partial equilibriums within the very microstate space. The most important one is that where each microstate is already in equilibrium with all its lower neighbour states. This property has already been observed in connection with multi-queue delay-loss systems, cf. |15,16,18|, revealing some interesting aspects as, e.g., with respect to

- proof-technique
- recursive calculation of the state probabilities
- approximations.

Especially the last aspect was very successful for approximate solutions of multidimensional state spaces yielding excellent accuracy, cf. |30,16,18| and Chapter 5 .

3.3.2 Characteristic Values

Any of the distributions eq.(27) - (30) yields the following results (index j refers to calls of group number j, quantities without index refer to Section 3.1) :

$$Y_j = A_j \tag{32}$$

$$\Omega_j = p(n;0) \cdot \frac{\varrho_j}{(1-\varrho)^2} = \frac{\varrho_j}{\varrho} \cdot \Omega \tag{33}$$

$$W_j = p(n;0) \cdot \frac{1}{1-\varrho} = W \tag{34}$$

$$w_j = p(n;0) \cdot \frac{h}{n(1-\varrho)^2} = w \tag{35a}$$

$$t_{W_j} = \frac{h}{n(1-\varrho)} = t_W \tag{35b}$$

$$W_j(>t) = W(>t) \quad , \; j = 1,2,\dots,g . \tag{36}$$

3.4 FULL ACCESS, FINITE SOURCE, MULTI-QUEUE MODEL M/M/n

Now the general delay system of Fig.1 is considered with g>1 input groups (queues), each having an individual number of sources q_j and idle source arrival rate α_j , $j = 1,2,\dots,g$. For short, only explicit solutions of state probabilities are given. The state description follows Section 3.3 .

For any of the disciplines FIFO, RANDOM, or LIFO with respect to all waiting calls holds:

$$\text{a)} \quad p(x_1,\dots,x_g;c_1,\dots,c_z) = p_0 \cdot \prod_{i=1}^{g}\left(\frac{\beta_i^{x_i}}{x_i!}\right) \cdot \prod_{j=1}^{z}\left(\frac{\beta_{c_j}}{n}\right) \cdot \prod_{k=1}^{g}\left(\prod_{\nu=0}^{x_k+z_k-1}(q_k-\nu)\right) \tag{37}$$

$$\text{b)} \quad p(x_1,\dots,x_g;z_1,\dots,z_g) = p_0 \cdot \prod_{i=1}^{g}\left(\frac{\beta_i^{x_i}}{x_i!}\right) \cdot z! \cdot \prod_{j=1}^{g} \frac{\left(\frac{\beta_j}{n}\right)^{z_j}}{z_j!} \cdot \prod_{k=1}^{g}\left(\prod_{\nu=0}^{x_k+z_k-1}(q_k-\nu)\right) , \tag{38}$$

where $\beta_i = \alpha_i / \varepsilon$. The equations can be proved as outlined in Section 3.3.1 . For FIFO eq.(37) was proved in |1|.

The marginal (macro) state probabilities analogously to eq.(29) - (30) are somewhat difficult to give in explicit form. Under a slight approximate assumption, however, an explicit solution can be derived in case of RANDOM and $\alpha_i = \alpha$, $i = 1,2,\dots,g$:

$$p(x;z_1,\dots,z_g) = \begin{cases} p_0 \cdot \dfrac{\beta^x}{x!} \cdot \prod\limits_{i=1}^{x-1}(Q-i) & , \; x < n \\ p_0 \cdot \dfrac{\beta^n}{n!} \cdot \prod\limits_{i=0}^{n-1}(Q-i) \cdot z! \cdot \prod\limits_{j=1}^{g} \dfrac{\left(\frac{\beta}{n}\right)^{z_j}}{z_j!} \cdot \prod\limits_{k=1}^{g}\left(\prod\limits_{\nu=0}^{z_k-1}\left(q_k\left[1-\frac{n}{Q}\right]-\nu\right)\right) , & x = n \end{cases} \tag{39}$$

where $\beta = \alpha/\varepsilon$, $Q = q_1+q_2+\dots+q_g$. To prove this formula, we assume for x = n that the n calls being in service had been generated uniformly from the various source groups, i.e. we assume $x_j \cong n \cdot q_j/Q$, $j = 1,2,\dots,g$. For g=1 this formula coincides with eq.(18).

3.5 LIMITED ACCESS, INFINITE SOURCE MODEL M/M/n(k)

Now, the n servers are assumed to be graded with limited accessibility k . Both, ideal and real gradings are considered.

3.5.1 Stationary State Probabilities

In the general case, there does not exist an explicit closed form solution. Exact results can only be obtained by iterative solution of equilibrium equations of microstate probabilities |16,18,19|. Here we only refer to approximate solutions for the symmetrical case $\lambda_j = \lambda/g$, $j=1,2,\dots,g$.

Defining the state (x,z) analogously to Section 3.3.1 , the state space acc. to Fig.4 is obtained.

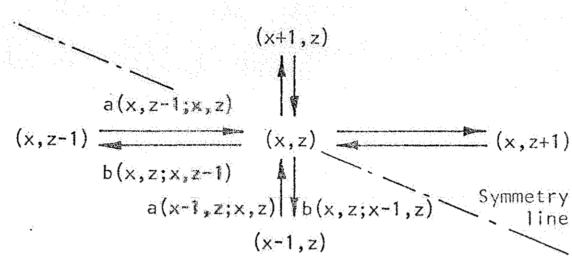


Fig.4: State space and transitions of the M/M/n(k) delay system with limited access.

In case of an infinite number of sources the transition rates are:

$$a(x-1,z;x,z) = \lambda \cdot u(x) \quad , \; x = 1,2,\dots,n \; , z \geq 0 \tag{40a}$$

$$a(x,z-1;x,z) = \lambda \cdot c(x) \quad , \; x = k,k+1,\dots,n \; , z \geq 1 \tag{40b}$$

$$b(x,z;x-1,z) + b(x,z;x,z-1) = x\varepsilon \quad , \; x = 1,2,\dots,n \tag{40c}$$

where c(x) the blocking probability and u(x) = 1-c(x) the passage probability of the grading acc. to

$$c(x) = \begin{cases} 0 & , \; x = 0,1,\dots,k-1 \\ \binom{x}{k} \Big/ \binom{n}{k} & , \; x = k,k+1,\dots,n \quad \text{(ideal gradings)} \\ \binom{x}{k^*} \Big/ \binom{n}{k^*} & , \; x = k,k+1,\dots,n \quad \text{(real gradings)} \end{cases} \tag{41}$$

In case of real gradings $c(x)$ is calculated using a reduced accessibility $k^* = k^*(n,k,\varrho,M,\text{interqueue disc.})$, e.g. in case of standard gradings |19|

$$k^* \cong k - \frac{n^2-k^2}{n^2}\left[\frac{k}{5}\varrho + \frac{k-3}{2}\cdot\varrho^{k/4} + a\cdot\frac{k}{5}\varrho^4\right]\cdot\frac{1}{M-1} \tag{42}$$

with a=0 or a=1 for interqueue discipline FIFO or RANDOM, respectively.

Under the special assumption of partial balance of state (x,z) with its lower neighbour states ("half-symmetry") the equilibrium state equation reads in the general case:

$$x\varepsilon\cdot p(x,z) = a(x-1,z;x,z)\cdot p(x-1,z) + a(x,z-1;x,z)\cdot p(x,z-1)\,. \tag{43}$$

By insertion of eq. (40a,b) and summation over all z the marginal state probabilities p(x) are obtained by recursion:

$$p(x) = p_0\cdot A^x\cdot\prod_{i=0}^{x-1}u(i)\Big/\prod_{i=1}^{x}\left[i-Ac(i)\right]\,,\quad x=0,1,\dots,n. \tag{44}$$

3.5.2 Characteristic Values

$$\Omega = \sum_{x=k}^{n}\sum_{z=0}^{\infty} z\cdot p(x,z) = \sum_{x=k}^{n} p(x)\cdot\sum_{i=k}^{x}\frac{Ac(i)}{i-Ac(i)} \tag{45}$$

$$W = \sum_{x=k}^{n}\sum_{z=0}^{\infty} p(x,z)\cdot c(x) = \sum_{x=k}^{n} p(x)\cdot c(x) \tag{46}$$

$$w = \Omega/\lambda\,,\qquad t_W = w/W\,. \tag{47a,b}$$

The df of waiting times can be approximated by a gamma df |19| or even better by a Weibull df analogously to eq. (15) and (16) using the following variation coefficient

$$c_W^2 \cong \begin{cases} \dfrac{8}{4+(1-\frac{k}{n})\cdot\varrho^3} - 1 & \text{for F/F} \\[2ex] \dfrac{4}{2-\varrho^{n/k}\cdot(1-M^{-2/3})} - 1 & \text{for R/F} \end{cases} \tag{48}$$

where F/F or R/F are short notations for interqueue/queue discipline FIFO (F) or RANDOM (R), respectively.

Eq. (43) - (47) were derived by M.Thierer |30,31| for ideal gradings ("Interconnection Delay Formula IDF"). Extensions with respect to real gradings and the df of waiting times were reported in |19|.

3.6 LIMITED ACCESS, FINITE SOURCE MODEL M/M/n(k)

In this section the solution technique from Section 3.5 is extended to the finite source model with $q_j = Q/g$ sources per grading group and $\alpha_j = \alpha$, $j=1,2,\dots,g$.

3.6.1 Stationary State Probabilities

With the state definition (x,z), $x=0,1,\dots,n$, $z=0,1,\dots,z_{max}(x)$, the same state diagramm of Fig.4 holds in the general case with

$$a(x-1,z;x,z) = \left[Qu(x-1)-(x-1)\left(1-d(x-1)\right)\right]\cdot\alpha\,,\quad \begin{matrix} x=1,2,\dots,n, \\ z=0,1,\dots,z_{max}(x)\end{matrix} \tag{49a}$$

$$a(x,z-1;x,z) = \left[Qc(x) - xd(x) - z+1\right]\cdot\alpha\,,\quad \begin{matrix} x=k,k+1,\dots,n, \\ z=1,2,\dots,z_{max}(x)\end{matrix} \tag{49b}$$

$$b(x,z;x-1,z) + b(x,z;x,z-1) = x\varepsilon\,,\quad x=1,2,\dots,n, \tag{49c}$$

where c(x) acc. to eq. (41), and d(x) that proportion of calls in service, originated by sources of momentary blocked grading groups. It was found |14| that d(x) can be approximated by

$$d(x) = g\cdot k\cdot c(x)/xM = n\cdot c(x)/x\,,\quad x=k,k+1,\dots,n. \tag{50}$$

Another complication is caused by the finiteness of the state space. Here, an upper number of waiting calls in state x is introduced by rounding up the expected number of sources not in service within the blocked grading groups :

$$z_{max}(x) = \begin{cases} 0 & ,\ x=0,1,\dots,k-1 \\ -\text{ENTIER}\left(-Qc(x)+xd(x)\right) & ,\ x=k,k+1,\dots,n. \end{cases} \tag{51}$$

The general equilibrium state equation reads analogously to eq. (43) with eq. (49), (50), and (51). The two-dimensional state probabilities p(x,z) are calculated numerically by recursion. From p(x,z) the state probabilities $\pi(x,z)$ seen by an arriving call are determined analogously to eq. (19):

$$\begin{aligned} \pi(x,0) &= (Q-x)\cdot p(x,0)/D\,, && x=0,1,\dots,k-1 \\ \pi(x,z) &= (Q-x-z)\cdot p(x,z)/D\,, && x=k,k+1,\dots,n \\ & && z=0,1,\dots,z_{max}(x)-1 \\ \pi(x,z_{max}(x)) &= \left[Qu(x)-x\cdot(1-d(x))\right]\cdot p(x,z_{max}(x))/D, && \end{aligned} \tag{52}$$

with

$$D = \sum_{i=0}^{k-1}(Q-i)p(i,0) + \sum_{i=k}^{n}\left[\sum_{j=0}^{z_{max}(i)-1}(Q-i-j)p(i,j) + \left[Qu(i)-i(1-d(i))\right]\cdot p(i,z_{max}(i))\right].$$

3.6.2 Characteristic Values

Based on p(x,z) and $\pi(x,z)$ the following values are derived:

$$Y = A = \sum_{x=1}^{n}\sum_{z=0}^{z_{max}(x)} x\cdot p(x,z) \tag{53}$$

$$\Omega = \sum_{x=k}^{n}\sum_{z=0}^{z_{max}(x)} z\cdot p(x,z) = Q - A - \frac{A}{\beta} \tag{54}$$

$$W = \sum_{x=k}^{n}\sum_{z=0}^{z_{max}(x)-1} \pi(x,z)\cdot\frac{Qc(x)-xd(x)-z}{Q-x-z} \tag{55}$$

$$w = h\cdot\Omega/A\,,\qquad t_W = w/\hat{W}\,. \tag{56a,b}$$

4. DELAY SYSTEMS WITH CONSTANT HOLDING TIMES

In this chapter single or multi-queue delay systems are considered having an infinite number of traffic sources with total arrival rate λ and constant holding time h .

4.1 FULL ACCESS, INFINITE SOURCE MODEL M/D/n

4.1.1 Stationary State Probabilities

For the exact analysis we follow the concept of describing the system state at time epochs t_0, t_0+h, t_0+2h, ... introduced by C.D.Crommelin |8|. This special view allows the description by an imbedded Markov chain. For conformity the system states at that time epochs will again be indicated by (x,z).

In the stationary state the following equilibrium equations hold:

$$p(x,0) = a_n\cdot q_x + \sum_{i=1}^{x} p(n,i)\cdot q_{x-i}\,,\quad x<n \tag{57a}$$

$$p(n,z) = a_n\cdot q_{n+z} + \sum_{i=1}^{n+z} p(n,i)\cdot q_{n+z-i}\,,\quad x=n,\ z\geq 0, \tag{57b}$$

where

$$a_x = \sum_{x=0}^{x} p(x,0)\,,\qquad q_i = \frac{A^i}{i!}\cdot e^{-A}\,. \tag{57c}$$

To solve eq. (57a,b) usually the generating function $G(z)$ of the stationary state probabilities $p(x,z)$ is defined. The solution for the state probabilities involves n roots of a transcendental equation.
For numerical reasons a direct method is used to solve eq. (57a,b) by iteration starting from an initial state distribution (M/M/n). An upper value z_{max} is determined such that $|\Omega(z_{max}) - \Omega|/\Omega < 10^{-8}$. To keep the compute time as low as possible a number of computational tricks had to be applied. By this method, systems with up to 200 servers and offered traffic per server up to 0.95 can be calculated relatively fast.

4.1.2 Characteristic Values

From the stationary state probabilities the characteristic mean values Ω, W, w, and t_W can be evaluated by application of definitions given in eq. (7) - (9).

The df of waiting times is explicitly known in case of FIFO with respect to all waiting calls, cf. |8| :

$$W(>t) = 1 - \sum_{\mu=0}^{r} \sum_{\nu=0}^{n-1} a_\nu \cdot \frac{\left[-A(\mu+\tau)\right]^{(r-\mu+1)n-1-\nu}}{\left[(r-\mu+1)n-1-\nu\right]!} \cdot e^{A(\mu+\tau)}, \tag{58}$$

where

$$t = (r+\tau)h \;, \quad 0 \leq \tau \leq 1.$$

The numerical evaluation, however, runs into complication for large delays. This problem can be overcome by programming with double-precision and by exponential continuation of $W(>t)$ for large t. The second moment can be determined using the general relationship between the k-th factorial moment of the number of calls in the system and the k-th ordinary moment of the waiting time for queuing models M/G/n |10|. Using this theorem, we arrive at

$$c^2_{W,FIFO} = \frac{W \cdot E\left[Z(Z-1)\right]}{\Omega^2} - 1 \tag{59}$$

where $E[Z \cdot (Z-1)]$ is easily obtained from the numerically explicit state probabilities. In case of $n=1$ the known result $c^2_{W,FIFO} = (2A+1)/3$ is obtained |29|.

In case of RANDOM with respect to all waiting calls, an exact solution is known only for $n=1$, cf. P.J.Burke |6|. In the multi-server case $n>1$ we approximate the second moment acc. to a heuristic similarity theorem |9,21| using the results of the second moment of waiting time for queuing systems M/M/n FIFO, M/M/n RANDOM, and M/D/n FIFO. By that

$$c^2_{W,RANDOM} \cong (2c^2_{W,FIFO} + \varrho)/(2-\varrho) \;, \tag{60}$$

where $c^2_{W,FIFO}$ acc. to eq. (59). This formula yields in case of $n = 1$ the exact result |29|. With t_W and $c_{W,RANDOM}$ an approximate expression for the df of waiting times is obtained using the Weibull df acc. to eq. (15), (16).

4.2 LIMITED ACCESS, INFINITE SOURCE MODEL M/D/n(k)

For this model M.Thierer |32| derived the stationary state equations based on the blocking probability $c(x)$ for ideal gradings and a two-dimensional state (x,z) as in Section 3.5 . Starting from that solution the method has been extended to real gradings and the df of waiting times.

4.2.1 Stationary State Probabilities

Acc. to |32| the equations of state are as follows:

$$p(x,0) = M(x,0) \cdot \sum_{j=0}^{x} q_{x-j} \cdot \sum_{i=0}^{n} p(i,j) \;, \quad x = 0,1,\ldots,n \tag{61a}$$

$$p(x,z) = M(x,z) \cdot \sum_{j=0}^{x+z} q_{x+z-j} \cdot \sum_{i=0}^{n} p(i,j) \;, \quad x = k,\ldots,n, \; z \geq 0, \tag{61b}$$

where

$$M(x,z) = M(x-1,z) \cdot u(x-1) + M(x,z-1) \cdot c(x) \;, \tag{61c}$$

with q_i acc. to eq. (57c) and $c(x)$ acc. to eq. (41) for ideal as well as real gradings (k^* acc. to eq. (42)).

For the numerical evaluation an iteration method is used as explained in Section 4.1.1 with initial values from the system M/M/n(k) and an upper value z_{max}. Using a number of computational tricks the necessary computer storage and compute time could be cut down such that fairly large systems with up to 200 servers and an offered traffic per server up to 0.95 can be calculated in less than 1 minute. The method was used for the calculation of tables |20|.

4.2.2 Characteristic Values

The characteristic values Ω, W, w, and t_W are calculated from the numerically explicit state probabilities $p(x,z)$ acc. to the definitions given in eq. (45) - (47).

For the df of waiting times only the case of RANDOM interqueue and FIFO queue discipline has been considered (R/F). For the variation coefficient of waiting times a heuristic formula was derived from systematic simulation studies:

$$c^2_{W,R/F} \cong \frac{3}{1.8 - \varrho^{n/k}(1 - M^{-2/3})} - 1 \tag{62}$$

for $n>k$, and $M \geq 2$. With t_W and $c_{W,R/F}$ the df of waiting times can be approximated by a Weibull df acc. to eq. (15) and (16).

5. DELAY-LOSS SYSTEMS WITH EXPONENTIAL HOLDING TIMES

In this chapter combined delay and loss systems are considered, i.e. systems with a limited number of waiting places acc. to Table 1 . For systems with general holding times, one server and finite waiting room (M/G/1-s) there were also exact and approximate solutions derived for the df of waiting times in case of FIFO and RANDOM queue discipline, cf. |20,21|, which are, however, not included in this paper.

5.1 FULL ACCESS, INFINITE SOURCE, SINGLE QUEUE MODEL M/M/n-s

5.1.1 Stationary State Probabilities

Using the same notations as in Section 3.1, the only difference in the calculation of state probabilities lies in the finiteness of the state space in Fig.3 with respect to $z_{max} = s$. The state probabilities are given by eq. (5a,b).

5.1.2 Characteristic Values

Analogously to Section 3.1:

$$Y = \sum_{x=0}^{n-1} x \cdot p(x,0) + n \cdot \sum_{z=0}^{s} p(n,z) = A(1-B) \tag{63}$$

$$\Omega = \sum_{z=0}^{s} z \cdot p(n,z) = p(n,0) \cdot \varrho \cdot \left[\frac{1-\varrho^s}{(1-\varrho)^2} - \frac{s\varrho^s}{1-\varrho}\right] \tag{64}$$

$$W = \sum_{z=0}^{s-1} p(n,z) = p(n,0) \cdot \frac{1-\varrho^s}{1-\varrho} \tag{65}$$

$$B = p(n,s) = p(n,0) \cdot \varrho^s \tag{66}$$

$$w = W \cdot \frac{h}{n} \left[\frac{1}{1-\varrho} - \frac{s\varrho^s}{1-\varrho^s}\right] \;, \quad t_W = w/W . \tag{67a,b}$$

The df of waiting times reads in case of FIFO

$$W(>t) = \frac{W}{1-\varrho^s} \left[\sum_{i=0}^{s-1} \frac{(\lambda t)^i}{i!} - \varrho^s \cdot \sum_{i=0}^{s-1} \frac{(n\varepsilon t)^i}{i!}\right] \cdot e^{-n\varepsilon t} . \tag{68}$$

These results go back to H.Störmer |27|. For the queue disciplines RANDOM and LIFO exact results for the df of waiting times and higher order moments have also been derived, cf. |17,18|. In the exact solution of the df of waiting times, however, eigenvalues of a system of differential equations are involved.

In case of RANDOM queue discipline approximate methods have been used acc. to the concept of approximating the conditional df of waiting time by exponential sums as explained in Section 3.2.2 for the calculation of tables |20|.

5.2 FULL ACCESS, FINITE SOURCE, SINGLE QUEUE MODEL M/M/n-s

5.2.1 Stationary State Probabilities

By the same arguments as used in Section 5.1 , the solution eq.(18a,b) holds for $p(x,z)$ also in case of a finite number of sources. The arriving call state distribution $\pi(x,z)$ is obtained analogously to eq.(19) by substituting the upper bound $(q-n)$ by s .

5.2.2 Characteristic Values

The characteristic values Y, Ω, W, w, t_W, and $W(>t)$ are obtained from eq.(20)-(24) by substituting $(q-n)$ by s in the upper summation bounds (in a delay-loss system holds $s < q-n$). The loss probability B is equal to $\pi(n,s)$. These results are included in the more general treatment of this model with priorities |4|.

In case of RANDOM queue discipline, the same principles can be applied for exact and approximate calculations as explained in Section 3.2 .

5.3 FULL ACCESS, INFINITE SOURCE, MULTI-QUEUE MODEL M/M/n-s

In the multi-queue delay-loss system two types of storage models will be distinguished: firstly, when there is a common storage capacity S for all calls, and secondly, when there is a limited storage capacity s_j exclusively for calls of group or class j , $j=1,2,..,g$.

5.3.1 Stationary State Probabilities

Using the identical state definitions as introduced in Section 3.3.1 we state:

For any of the disciplines FIFO, RANDOM, LIFO with respect to all waiting calls the state distributions acc. to eq.(27) - (31) hold also in case of delay-loss systems with common storage capacity S for all classes of calls ($z \leq S$).

Such a general theorem does not apply in case of individual storages per class. Nevertheless, it can be proved:

For any of the disciplines FIFO, RANDOM, LIFO with respect to all waiting calls the marginal state distributions acc. to eq.(29),(30) hold in case of delay-loss systems with an individual storage capacity s_j for calls of class j ($z_j \leq s_j$) , $j=1,2,..,g$.

The proof of these theorems again is done by inserting the results into the equilibrium state equations and stating consistency. In case of RANDOM eq.(30) has been proved directly |15|.

For the practical case where $s_j = s$ and $A_j = A/g$, a special solution has been derived which is included in an algorithm given in Section 5.5 .

5.3.2 Characteristic Values

Dependent on the underlying storage model the characteristic values can be defined straightforwardly from the state distribution eq.(30). The definitions will not be reproduced here.

5.4 FULL ACCESS, FINITE SOURCE, MULTI-QUEUE MODEL M/M/n-s

As in Section 3.4, we will only concentrate on the state probabilities. Using the same state definitions as introduced in Section 3.3.1 , we find for the two considered storage models:

For any of the disciplines FIFO, RANDOM, LIFO with respect to all waiting calls the state distributions acc. to eq.(37),(38) also hold in case of delay-loss systems with common storage capacity S for all classes of calls ($z \leq S$).

Under the same approximate assumption as in Section 3.4 eq.(39) holds also in case of delay-loss systems with individual storage capacity s_j for calls of class j ($z_j \leq s_j$), $j=1,2,..,g$.

5.5 LIMITED ACCESS, INFINITE SOURCE MODEL M/M/n(k)-s

The exact analysis of graded delay-loss systems has been carried out under general assumptions by numerical means, cf. |16,18|. Here, we only confine o.s. to symmetrical systems having $s_j = s$ waiting places and equal arrival rates $\lambda_j = \lambda/g$ per grading group, $j=1,2,..,g$.

5.5.1 Stationary State Probabilities

Contrary to the state descriptions above, we define a state $(x ; u_1,..,u_s)$ with

- x total number of calls in service
- u_j total number of calls waiting on the j-th waiting place, $j=1,2,..,s$.

This description by "waiting place rows" reduces the number of states considerably |16|. The state space and its transitions are shown for the general case in Fig.5 .

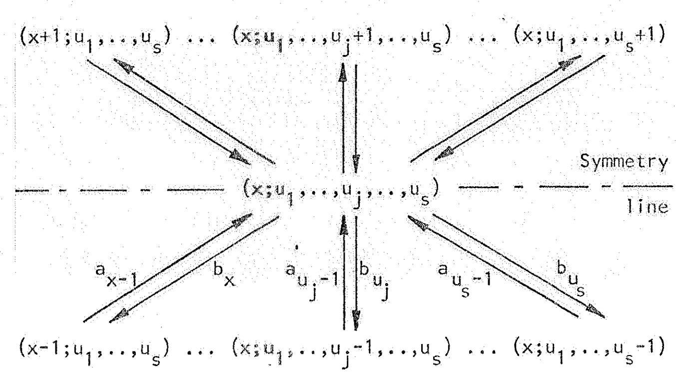


Fig.5: State space and transitions of the M/M/n(k)-s delay-loss system with limited access.

With the general transition rates

$$a_{x-1} = \lambda\left[1-c(x-1)\right] \tag{69a}$$

$$a_{u_1-1} = \lambda\left[gc(x)-(u_1-1)\right]/g \tag{69b}$$

$$a_{u_j-1} = \lambda\left[u_{j-1}-(u_j-1)\right]/g \;, \quad j=2,3,\dots,s \tag{69c}$$

$$b_x + b_{u_1} + \dots + b_{u_s} = x\varepsilon \tag{69d}$$

and a partial equilibrium assumption (cf. Section 3.3.1) the following recursion formula can be derived:

$$x\varepsilon \cdot p(x;u_1,\dots,u_s) = a_{x-1}\cdot p(x-1;u_1,\dots,u_s) + \sum_{j=1}^{s} a_{u_j-1}\cdot p(x;\dots,u_j-1,\dots) \tag{70}$$

with

$$u_s \leq u_{s-1} \leq \dots \leq u_1 \leq -\mathrm{ENTIER}\left(-gc(x)\right) \;, \tag{71}$$

and c(x) acc. to eq.(41). For real gradings the value k^* for pure delay systems |18| has to be modified by substituting $\rho=A/n$ by Y/n and by inserting a reduction term $s/(s+1)$; for standard gradings, e.g., we have

$$k^* \cong k - \frac{n^2-k^2}{n^2}\cdot\left[\frac{k}{5}\cdot\left(\frac{Y}{n}\right) + \frac{k-3}{2}\cdot\left(\frac{Y}{n}\right)^{k/4} + a\cdot\frac{k}{5}\cdot\left(\frac{Y}{n}\right)^4\right]\cdot\frac{1}{M-1}\cdot\frac{s}{s+1}, \quad s \geq 1, \tag{72}$$

insted of eq.(42). Since Y is a result, iterations have to be performed |14|. Finally, we state that eq.(70) yields the exact results in case of full access.

5.5.2 Characteristic Values

$$Y = \sum_{x=0}^{n}\sum_{u_1=0}^{g}\dots\sum_{u_s=0}^{g} x\,p(x;u_1,\dots,u_s) = A(1-B) \tag{73}$$

$$\Omega = \sum_{x=k}^{n}\sum_{u_1=0}^{g}\dots\sum_{u_s=0}^{g} (u_1+\dots+u_s)\cdot p(x;u_1,\dots,u_s) \tag{74}$$

$$W = \sum_{x=k}^{n}\sum_{u_1=0}^{g}\dots\sum_{u_s=0}^{g} \left[c(x)-\frac{u_s}{g}\right]\cdot p(x;u_1,\dots,u_s) \tag{75}$$

$$B = \sum_{x=k}^{n}\sum_{u_1=0}^{g}\dots\sum_{u_s=0}^{g} \frac{u_s}{g}\cdot p(x;u_1,\dots,u_s) \tag{76}$$

$$w = \Omega/\lambda, \qquad t_W = w/W. \tag{77a,b}$$

5.6 LIMITED ACCESS, FINITE SOURCE MODEL M/M/n(k)-s

5.6.1 Stationary State Probabilities

The method of Section 5.5 has been extended to a finite number of $q_j = q = Q/g$ sources per grading group with $\alpha_j = \alpha$, $j=1,2,\dots,g$. The transition rates for the state space acc. to Fig.5 are:

$$a_{x-1} = \left[Q\,u(x-1) - (x-1)\left(1-d(x-1)\right)\right]\cdot\alpha \tag{78a}$$

$$a_{u_1-1} = \left[g\,c(x) - (u_1-1)\right]\cdot\left(q - \frac{x\,d(x)}{g\,c(x)}\right)\cdot\alpha \tag{78b}$$

$$a_{u_j-1} = \left[u_{j-1} - (u_j-1)\right]\cdot\left(q-(j-1)-\frac{x\,d(x)}{g\,c(x)}\right)\cdot\alpha, \quad j=2,3,\dots,s \tag{78c}$$

$$b_x + b_{u_1} + \dots + b_{u_s} = x\varepsilon \tag{78d}$$

with c(x) acc. to Section 5.5 and d(x) acc. to eq.(50). Analogously to Section 5.5.1 the state probabilities are obtained from eq.(70) with transition rates acc. to eq.(78a-d). From $p(x;u_1,\dots,u_s)$ the arriving call state distribution $\pi(x;u_1,\dots,u_s)$ is found:

$$\pi(x;u_1,\dots,u_s) = \frac{(Q-x-u_1-\dots-u_s)\cdot p(x;u_1,\dots,u_s)}{\sum_{i=0}^{n}\sum_{j_1=0}^{g}\dots\sum_{j_s=0}^{g}(Q-i-j_1-\dots-j_s)\cdot p(i;j_1,\dots,j_s)}. \tag{79}$$

5.6.2 Characteristic Values

The carried traffic Y, the mean number of waiting calls Ω, and the mean waiting times t_W and w follow directly acc. to definitions given in eq.(73), (74), and (77a,b). For the probabilities of delay W and of loss B holds

$$W = \sum_{x=k}^{n}\sum_{u_1=0}^{g}\dots\sum_{u_s=0}^{g}\frac{Q\,c(x)-x\,d(x)-u_1-\dots-u_s-u_s\left[q-\frac{x\,d(x)}{g\,c(x)}-s\right]}{Q-x-u_1-\dots-u_s}\,\pi(x;u_1,\dots,u_s) \tag{80}$$

$$B = \sum_{x=k}^{n}\sum_{u_1=0}^{g}\dots\sum_{u_s=0}^{g} u_s\left[q-\frac{x\,d(x)}{g\,c(x)}-s\right]\cdot\pi(x;u_1,\dots,u_s). \tag{81}$$

Finally, we note for reasons of completeness that the corresponding solutions for graded loss systems are reported in |5|,|23|,|24|,|11|, and |2| both for ideal and real gradings.

6. NUMERICAL RESULTS

In this chapter some calculated results are given to show the influence of various system and traffic parameters on the main characteristic values, and to check the accuracy of approximate results by comparison with simulations.

6.1 FULL ACCESS, SINGLE QUEUE MODELS M/M/n

The mean values of the single queue delay system M/M/n with full access and finite or infinite number of sources are known exactly. In Fig.6 the dfs of waiting times $W(>\tau)/W$ are given for systems with n=10 servers, q=20 and $q \to \infty$ traffic sources, and service acc. to the queue disciplines FIFO (dashed curves) and RANDOM (solid curves), respectively. The results for FIFO hold exactly whereas the results for RANDOM are approximations acc. to Sections 3.1.2 and 3.2.2 . The approximate results are in good accordance with simulations.

6.2 LIMITED ACCESS MODELS M/M/n(k)

Fig.7 presents the probability of delay W and the relative mean waiting time of waiting calls $\tau_W = t_W/h$ dependent on the offered traffic per server A/n for a limited access delay system with n=30 and k=10. The solid curves and the simulation results hold for a standard grading with $M=3^1/_3$, q=15 sources per group, and R/F discipline. For comparison, the corresponding results are given for this grading for $q \to \infty$ and R/F discipline (dashed curves) as well as in case of an ideal grading with a total of Q = 150 sources (dotted curves).

6.3 LIMITED ACCESS MODELS M/D/n(k)

In Fig.8 the characteristic values W and τ_W are drawn versus A/n for gradings with n=30, k=10. The solid curves and simulations hold in case of a standard grading with M=2 , R/F discipline, and constant holding time. For comparison the corresponding results are given for this grading with exponential holding time (dashed curves) and for an ideal grading with constant holding time (dotted curves).

Note, that the holding time characteristic influences mainly τ_W whereas W remains nearly unchanged (real gradings).

6.4 FULL ACCESS, SINGLE QUEUE MODEL M/D/n

In Fig.9 the dfs of waiting times are shown for the delay system M/D/n with n=10 for FIFO (dashed curves) and RANDOM (solid curves) queue disciplines, respectively. In case of RANDOM, a two-moment approximation has been used acc. to Section 4.1.2 . Compared with simulations, eq.(60) generally yields a slightly too high variation coefficient which is also reflected in Fig.9 .

6.5 LIMITED ACCESS MODELS M/M/n(k)-s

Fig. 10 and 11 give results for the limited access delay-loss system with standard grading having n=30, k=10, M=2, and s=3 waiting places per grading group. In Fig.10 an infinite source traffic is assumed. For comparison, the corresponding pure delay system results are added (dashed curves). Fig.11 presents the corresponding results for the finite source model with a total of Q=120 traffic sources. For both, the infinite and finite source delay-loss system with limited access, the calculated results are sufficiently accurate for applications.

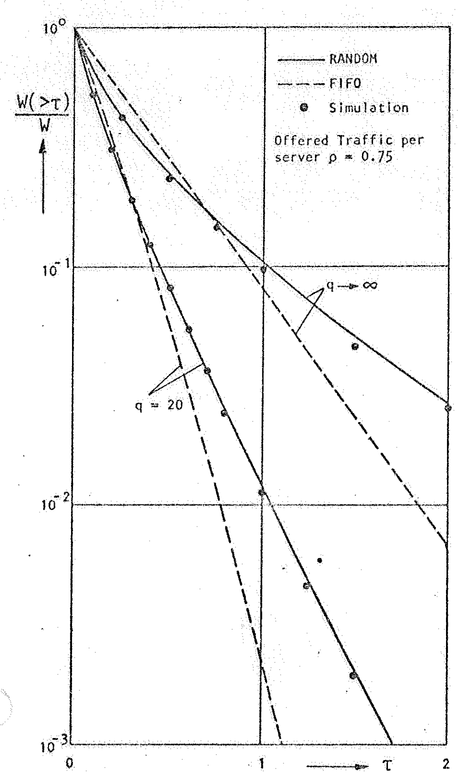


Fig.6: Distribution function of waiting times of waiting calls W(>τ)/W versus normalized time τ=t/h

Full access delay systems M/M/n
n = k = 10, g = 1
Number of sources: q = 20,∞
Queue disciplines: FIFO, RANDOM

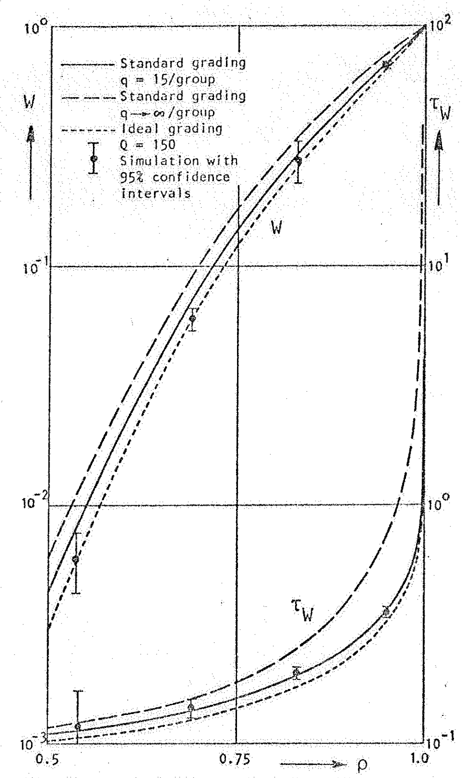


Fig.7: Probability of delay W and rel. mean waiting time of waiting calls τ_W versus offered traffic per server ρ = A/n

Limited access delay systems M/M/n(k)
n=30, k=10, M=$3^1/_3$ (standard grading)
Number of sources per group: q = 15,∞
Interqueue/queue discipline: R/F

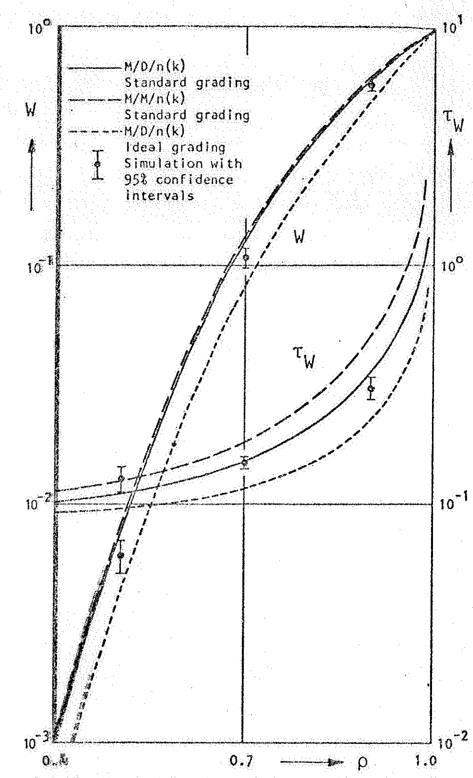
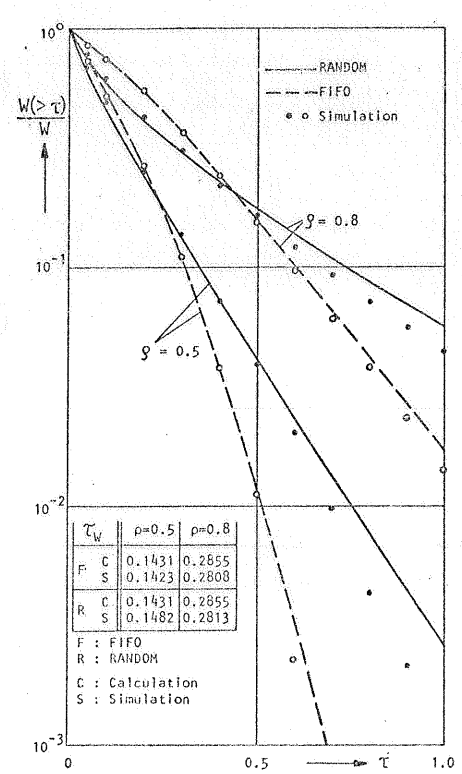


Fig.8: Probability of delay W and rel. mean waiting time of waiting calls τ_W versus offered traffic per server ρ= A/n

Limited access delay systems M/D/n(k) and M/M/n(k)
n=30, k=10, M=2 (standard grading)
Number of sources per group q → ∞
Interqueue/queue discipline: R/F



τ_W		ρ=0.5	ρ=0.8
F	C	0.1431	0.2855
	S	0.1423	0.2808
R	C	0.1431	0.2855
	S	0.1482	0.2813

Fig.9: Distribution function of waiting times of waiting calls W(>τ)/W versus normalized time τ=t/h

Full access delay systems M/D/n
n = k = 10, g = 1
Number of sources: q → ∞
Queue disciplines: FIFO, RANDOM

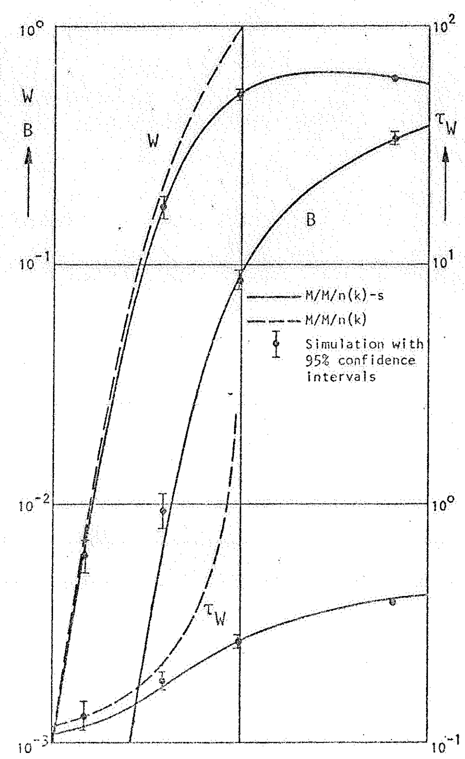


Fig.10: Probability of delay W, probability of loss B, and rel. mean waiting time of waiting calls τ_W versus ρ= A/n

Lim.access delay-loss system M/M/n(k)-s
n=30, k=10, M=2 (standard grading)
Number of waiting places/group: s=3,∞
Number of sources per group: q → ∞
Interqueue/queue discipline: R/F

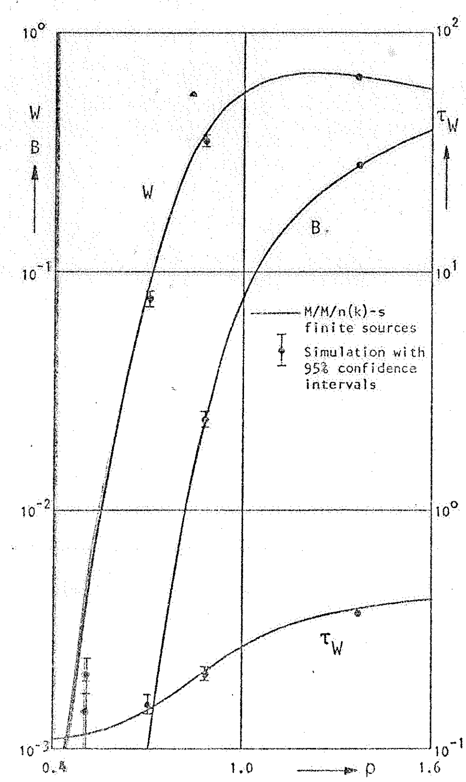


Fig.11: Probability of delay W, probability of loss B, and rel. mean waiting time of waiting calls τ_W versus ρ=A/n

Lim.access delay-loss system M/M/n(k)-s
n=30, k=10, M=2 (standard grading)
Number of waiting places/group: s=3
Number of sources per group: q=20
Interqueue/queue discipline: R/F

CONCLUSION

In this paper a number of single-stage delay systems has been reviewed in a systematical way. Starting from known results, which are either briefly repeated or referred to, various extensions have been presented with respect to multi-queues (classes), finite number of traffic sources, limited accessibility, and different queue disciplines. The results were partly used for tables on delay systems and form basic modules for the analysis of multi-stage connecting arrays (link systems) with waiting. The extensions in case of multi-class delay systems are also applicable to Markovian queuing networks with respect to queue disciplines RANDOM and LIFO. In case of multi-class delay-loss systems two subcases of limited storage capacity are distinguished: common storage for all classes or individual storage per class. The general product solutions of pure delay systems can partially be extended to those delay-loss systems. All presented solutions are adequate for a computational evaluation; approximate solutions were checked by simulations yielding sufficient accuracy for practical applications (dimensioning).

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