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# Graded Delay Systems with Infinite or Finite Source Traffic and Exponential or Constant Holding Time

**Part I: Theory**

by Gerald Kampe and Paul Kühn

# Graded Delay Systems with Infinite or Finite Source Traffic and Exponential or Constant Holding Time\*

## Part I: Theory

by Gerald Kampe\*\* and Paul Kühn\*\*

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The paper deals with single-stage delay systems of the types  $M/M/n$  and  $M/D/n$  and single-stage delay-loss systems of the type  $M/M/n - s$  both in case of full or limited accessibility. Further distinctions are made between single and multi-queues, ideal and real gradings, and a finite or infinite number of traffic sources. Waiting calls are served according to the disciplines FIFO, RANDOM or LIFO.

In Part I of the paper, the theoretical background is developed. The various queuing systems are reviewed systematically with respect to their stationary state probabilities and characteristic traffic values (means, distribution functions of waiting time).

Part II concludes with numerical results of the major characteristic values and simulation results for validation of approximate analysis methods.

### Einstufige Wartesysteme mit vollkommener oder unvollkommener Erreichbarkeit, unendlicher oder endlicher Quellenzahl und exponentiellen oder konstanten Belegungsauern — Teil I: Theorie

Der Beitrag behandelt einstufige Wartesysteme des Typs  $M/M/n$  und  $M/D/n$  bzw. einstufige Warte-Verlustsysteme des Typs  $M/M/n - s$  jeweils im Fall vollkommener und unvollkommener Erreichbarkeit. Weitere Unterscheidungsmerkmale der betrachteten Systeme sind die Anzahl der Warteschlangen, ideale oder reale Mischungen sowie eine endlich oder unendlich große Anzahl von Verkehrsquellen. Zur Abfertigung wartender Rufe werden die Disziplinen FIFO, RANDOM und LIFO vorausgesetzt.

Im Teil I des Beitrags werden die theoretischen Grundlagen entwickelt. Die verschiedenen Warteschlangenmodelle werden systematisch behandelt hinsichtlich ihrer stationären Zustandswahrscheinlichkeiten und charakteristischen Verkehrswerte (Mittelwerte, Verteilungsfunktionen der Wartezeiten).

Teil II schließt den Beitrag ab mit numerischen Ergebnissen der wichtigsten Verkehrsgrößen sowie Simulationsergebnissen zum Genauigkeitsnachweis von approximativem Analyseverfahren.

## 1. Introduction

### 1.1. Problem

In certain types of telephone and data switching systems the data traffic flow from peripheral to centralized devices is handled by concentrating switching networks. Usually, these access networks are operated as delay systems. To design such networks, reliable methods for grade-of-service calculation are necessary.

To describe the traffic flow in such systems, an adequate queuing model is constructed consisting of servers and waiting places for queuing. Arriving requests (calls) are generated by a finite or an infinite number of traffic sources. If there are multiple servers, a certain server is selected acc. to a hunting discipline. If all accessible servers are occupied, an arriving call may wait if there is a waiting place available. Waiting calls are selected for service according to a certain interqueue or queue discipline.

Input and service processes of the queuing model are described by distribution functions (df) of inter-arrival times of calls and service times, respectively.

The queuing model is analyzed by considering the discrete stochastic process of the random number of calls within the system. Under the assumption of stationarity the steady-state probabilities are obtained from which other characteristic values are derived for system dimensioning as e.g. mean waiting times.

In the past a number of solutions has been derived for single-stage delay systems. These solutions will now be extended with respect to criteria as finite sources, real gradings or limited waiting capacity, respectively. The results are part of a table book on delay systems [20] and are also used as basic modules for the analysis of link systems with waiting [13].

### 1.2. Survey of analyzed queuing models

To represent the queuing models briefly, Kendall's short notation will be used with some modifications as follows:

$$X/Y/n(k) - s$$

where

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$X$  type of arrival process, e.g. M: Markovian,  
 $Y$  type of service process, e.g. D: Deterministic,  
 $n$  number of servers,  
 $k$  accessibility (if limited),  
 $s$  number of waiting places (if limited).

With this notation an overview on the problems dealt in this paper is given in Table 1.

Table 1. Survey of queuing models ( $q$  number of traffic sources).

Type of queuing model	Delay systems		Delay-loss systems	
	$M/M/n(k)$ $q \rightarrow \infty$	$M/D/n(k)$ $q < \infty$	$M/M/n(k)-s$ $q \rightarrow \infty$	$M/M/n(k)-s$ $q < \infty$
	Chapter 3	Chapter 4	Chapter 5	
Full access	single queue	3.1 3.2	4.1	5.1 5.2
	multi-queue	3.3 3.4		5.3 5.4
Limited access	ideal grading	3.5 3.6	4.2	5.5 5.6
	real grading			

## 2. General Queuing Model

### 2.1. System structure and operating modes

The system structure of the general queuing model is shown in Fig. 1.

Arriving calls hunt the servers with full or limited accessibility, cf. Fig. 2.

The special interconnection scheme in case of limited access groups is called "grading". Two main

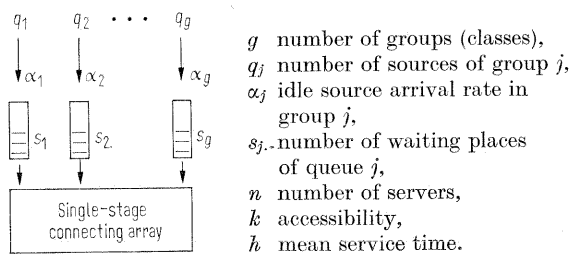


Fig. 1. General queuing model.

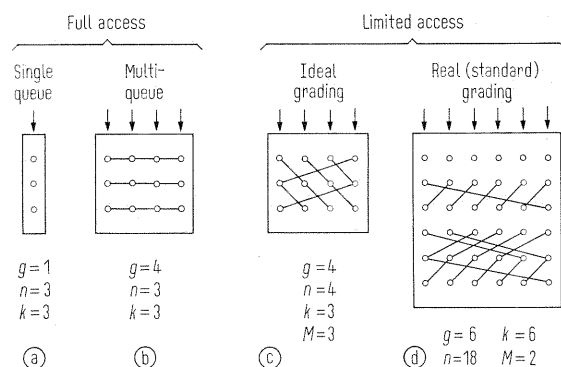


Fig. 2. Examples of connecting arrays with full or limited access.

types will be distinguished:

- "ideal" gradings with a combinatorial number of grading groups  $g = \binom{n}{k}$  in case of random hunting or  $g = n \binom{n}{k} k!$  in case of sequential hunting, and
- nonideal ("real") gradings.

In the latter case only standard gradings [11] will be considered. From  $n, k$ , and  $g$  the mean interconnecting number is defined as  $M = g \cdot k/n$ .

Arriving calls are handled according to certain operating strategies comprising hunting mode, queue and interqueue disciplines. As hunting mode we assume random or sequential hunting. For nonideal gradings only sequential hunting will be regarded. If no idle server is accessible, the incoming call occupies a waiting place in its group; if all available waiting places are occupied, the incoming call is lost.

Waiting calls are selected for service according to operating modes as interqueue disciplines (selection of a certain queue in a multi-queue system), queue disciplines (selection of a call within a queue), or disciplines with regard to all waiting calls in a multi-queue system. In this paper the basic disciplines FIFO (first-in, first-out), RANDOM, and LIFO (last-in, first-out) will be considered. When a call is selected for service, its waiting place is released immediately.

### 2.2. Arrival and service processes

Arriving calls are generated either by a finite number  $q$  of traffic sources with an idle source arrival rate  $\alpha$  or by an infinite number  $q \rightarrow \infty$  of traffic sources with a constant arrival rate  $\lambda$ . Throughout the paper only negative exponential df's are assumed for the random interarrival times  $T_A$ , i.e.

$$A(t) = P\{T_A \leq t\} = \begin{cases} 1 - e^{-\lambda t}, & q \rightarrow \infty \\ 1 - e^{-(q-r)xt}, & q < \infty \end{cases} \quad (1)$$

where  $r$  the number of nonidle sources. The random service (holding) times  $T_H$  are mutual independent and identically distributed according to a negative exponential or constant df with mean  $h$ , i.e.

$$H(t) = P\{T_H \leq t\} = \begin{cases} 1 - e^{-\varepsilon t}, & \text{Markovian (M)} \\ 0, t < h, & \text{Deter-} \\ 1, t \geq h, & \text{ministic (D)} \end{cases} \quad (2)$$

where  $\varepsilon = 1/h$  the termination rate. The service time is independent of the source group by which a call has been generated. From  $\lambda, \alpha$ , and  $h$  the offered traffic  $A = \lambda \cdot h$  and the idle source offered traffic  $\beta = \alpha \cdot h$  are defined in case of infinite or finite number of sources, respectively.

### 2.3. Characteristic values

Characteristic values are measures for the system performance (grade-of-service). They are derived from random variables describing the stochastic behaviour of the service systems as, e.g.,  $X$  (random number of calls in service),  $Z$  (random number of calls waiting), or  $T_W$  (random waiting time);

- $p(\dots)$  stationary state probabilities seen from an outside observer,
- $\pi(\dots)$  stationary state probabilities seen from an arriving call,
- $Y$  carried traffic,
- $\Omega$  mean queue length of waiting calls,
- $W$  probability of delay,
- $B$  probability of loss,
- $w$  mean waiting time of offered calls,
- $t_W$  mean waiting time of waiting calls,
- $W(>t)$  complementary df of waiting times.

**3. Delay Systems with Exponential Holding Times**

In this chapter pure delay systems will be considered having single or many queues, infinite or finite number of traffic sources, and full or limited accessibility, respectively.

**3.1. Full access, infinite source, single queue model  $M/M/n$**

**3.1.1. Stationary state probabilities**

To describe the system state, a single state variable  $i = 0, 1, \dots$  is sufficient indicating the number of calls in the system. For reasons of conformity in this paper a distinction is made between the numbers  $x$  and  $z$  of calls being served or waiting by introducing a pseudo-twodimensional state variable  $(x, z)$ , cf. Fig. 3.

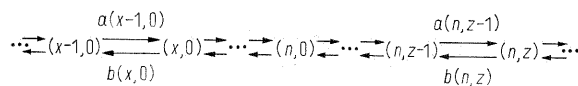


Fig. 3. State space and transitions of the  $M/M/n$  delay system with full access and a single queue.

In case of an infinite source traffic, the transition rates are

$$\begin{aligned} a(x, 0) &= \lambda, & x &= 0, 1, \dots, n-1, \\ a(n, z) &= \lambda, & z &= 0, 1, \dots, \end{aligned} \quad (3a)$$

$$\begin{aligned} b(x, 0) &= x\varepsilon, & x &= 1, 2, \dots, n-1, \\ b(n, z) &= n\varepsilon, & z &= 0, 1, \dots \end{aligned} \quad (3b)$$

In the stationary state the equilibrium equations for the probabilities of state  $p(x, z)$  are

$$p(x, 0) b(x, 0) = p(x-1, 0) a(x-1, 0), \quad (4a)$$

$$x = 1, 2, \dots, n,$$

$$p(n, z) b(n, z) = p(n, z-1) a(n, z-1), \quad (4b)$$

$$z = 1, 2, \dots$$

Introducing eqs. (3a, b) into eqs. (4a, b) we have

$$p(x, 0) = p_0 \frac{A^x}{x!}, \quad x \leq n, \quad (5a)$$

$$p(n, z) = p_0 \frac{A^n}{n!} \varrho^z, \quad z \geq 0, \quad (5b)$$

where  $p_0 = p(0, 0)$  and  $\varrho = A/n$ .

The probability  $p_0$  indicates generally that the system is in the idle state; it is found from the normalizing condition that the sum of all state probabilities must equate to one.

**3.1.2. Characteristic values**

The characteristic values are defined as follows:

$$Y = E[X] = \sum_{x=0}^{n-1} x p(x, 0) + n \sum_{z=0}^{\infty} p(n, z) = A, \quad (6)$$

$$\Omega = E[Z] = \sum_{z=0}^{\infty} z p(n, z) = p(n, 0) \frac{\varrho}{(1-\varrho)^2}, \quad (7)$$

$$W = P\{T_W > 0\} = \sum_{z=0}^{\infty} p(n, z) = p(n, 0) \frac{1}{1-\varrho}, \quad (8)$$

$$w = E[T_W] = \Omega/\lambda = p(n, 0) \frac{h}{n(1-\varrho)^2}, \quad (9a)$$

$$t_W = E[T_W | T_W > 0] = \frac{\Omega}{\lambda W} = \frac{h}{n(1-\varrho)}. \quad (9b)$$

The df of waiting times depends on the underlying queue discipline. Generally it holds

$$W(>t) = P\{T_W > t\} = \sum_{z=0}^{\infty} p(n, z) w(t|z), \quad (10)$$

where  $w(t|z)$  the conditional complementary df of waiting times conditioned on the number  $z$  of waiting calls met by an arriving call. In case of FIFO

$$w(t|z) = P\{T_W > t|z\} = \sum_{i=0}^z \frac{(n\varepsilon t)^i}{i!} e^{-n\varepsilon t},$$

$$z = 0, 1, \dots \quad (11a)$$

which results to

$$\frac{W(>t)}{W} = e^{-t/t_W}. \quad (11b)$$

So far, solutions (5) to (11) go back to A. K. Erlang [5].

For the RANDOM queue discipline, the values  $w(t|z)$  are solutions of a differential equation system formulated by C. Palm and E. Vaulot [7], [25], [28] for the random process of waiting (Kolmogorov-backward equations):

$$w'(t|z) = -(\lambda + n\varepsilon) w(t|z) + \lambda w(t|z+1) + n\varepsilon \frac{z}{z+1} w(t|z-1), \quad z \geq 0. \quad (12)$$

To evaluate the df of waiting times numerically, approximations are important based on the lower moments, viz.

$$E[T_W | T_W > 0] = h/n(1-\varrho) = t_W, \quad (13a)$$

$$E[T_W^2 | T_W > 0] = 2h^2/n^2(1-\varrho)^2(1-\varrho/2), \quad (13b)$$

$$E[T_W^3 | T_W > 0] = 6h^3(4+2\varrho)/n^3(1-\varrho)^3(2-\varrho)^2, \quad (13c)$$

respectively the variation coefficient  $c_W$  according to

$$c_W^2 = \frac{E[T_W^2 | T_W > 0]}{E[T_W | T_W > 0]^2} - 1 = \frac{2-\varrho}{2+\varrho}. \quad (13d)$$

J. Riordan [25] approximates  $W(>t)$  by a hyper-exponential df

$$\frac{W(>t)}{W} \approx p e^{-a_1 t} + (1-p) e^{-a_2 t} \quad (14)$$

where the parameters  $p, a_1, a_2$  are determined from the first, second, and third moment according to eqs. (13a-c). A new approximation [20], [21] is based on the Weibull df:

$$\frac{W(>t)}{W} \approx e^{-(at)^b} \quad (15)$$

where the parameters  $a$  and  $b$  are determined only from the first and second moment by iteration according to

$$t_W = \Gamma\left(1 + \frac{1}{b}\right) / a, \quad (16a)$$

$$c_W^2 = \frac{\Gamma\left(1 + \frac{2}{b}\right)}{\Gamma\left(1 + \frac{1}{b}\right)^2} - 1. \quad (16b)$$

This approximation fits better with simulation than eq. (14) although it is based on two moments only!

### 3.2. Full access, finite source, single queue model $M/M/n$

#### 3.2.1. Stationary state probabilities

In a finite source pure delay system with a total number of  $q$  sources there are  $s = q - n$  waiting places available. The state space is similar to Fig. 3 but limited by  $z \leq z_{\max} = q - n$ . The transition rates are:

$$a(x, 0) = (q - x)\alpha, \quad x = 0, 1, \dots, n - 1, \quad (17a)$$

$$a(n, z) = (q - n - z)\alpha, \quad z = 0, 1, \dots, q - n - 1,$$

$$b(x, 0) = x\varepsilon, \quad x = 1, 2, \dots, n - 1, \quad (17b)$$

$$b(n, z) = n\varepsilon, \quad z = 0, 1, \dots, q - n.$$

In the stationary state the equilibrium equations for the state probabilities  $p(x, z)$  seen from an outside observer are given by eqs. (4a, b) with finite state space ( $z_{\max} = q - n$ ). Introducing eqs. (17a, b) into eqs. (4a, b) we find

$$p(x, 0) = p_0 \frac{\beta^x}{x!} \prod_{i=0}^{x-1} (q - i), \quad x \leq n, \quad (18a)$$

$$p(n, z) = p_0 \frac{\beta^n}{n!} \left(\frac{\beta}{n}\right)^z \prod_{i=0}^{n+z-1} (q - i), \quad z \geq 0. \quad (18b)$$

An arriving call meets a state  $(x, z)$  with probability

$$\pi(x, z) = \frac{(q - x - z) p(x, z)}{\sum_{i=0}^{n-1} (q - i) p(i, 0) + \sum_{j=0}^{q-n} (q - n - j) p(n, j)}, \quad x = 0, 1, \dots, n, \quad z = 0, 1, \dots, q - n - 1. \quad (19)$$

#### 3.2.2. Characteristic values

Analogously to Section 3.1.2 it holds

$$Y = A = \sum_{x=0}^{n-1} x p(x, 0) + n \sum_{z=0}^{q-n} p(n, z), \quad (20)$$

$$\Omega = \sum_{z=0}^{q-n} z p(n, z) = q - A - \frac{A}{\beta}, \quad (21)$$

$$W = \sum_{z=0}^{q-n-1} \pi(n, z), \quad (22)$$

$$w = \frac{\Omega}{A} h, \quad t_W = \frac{w}{W}. \quad (23a, b)$$

The df of waiting times is determined by

$$W(>t) = \sum_{z=0}^{q-n-1} \pi(n, z) w(t|z). \quad (24)$$

In case of FIFO  $w(t|z)$  is given by eq. (11a). Solutions (18) to (24) were derived by F. L. Bauer and H. Störmer [3]. An approximate solution was reported by A. Lotze [22].

For RANDOM service of waiting calls a system of differential equations has to be solved analogously to eq. (12). The equations read in the general case of  $z$ :

$$w'(t|z) = -[(q - n - z - 1)\alpha + n\varepsilon] w(t|z) + (q - n - z - 1)\alpha w(t|z + 1) + n\varepsilon \frac{z}{z + 1} w(t|z - 1). \quad (25)$$

The solution of eq. (25) incorporates the roots of a characteristic equation [26]. For numerical purposes we use the concept of approximating the conditional df  $w(t|z)$  by exponential or hyperexponential df meeting the first, second, and third moment of  $w(t|z)$  exactly. This concept was introduced in case of systems  $M/M/n - s$  with RANDOM queue discipline and yields extremely high accuracy [18]. By this

$$w(t|z) \approx p_z e^{-a_{1z}t} + (1 - p_z) e^{-a_{2z}t} \quad (26)$$

where the parameters  $p_z, a_{1z}, a_{2z}$  are determined from the first, second, and third moment of  $w(t|z)$ . These moments are obtained from linear systems of equations derived from eq. (25), [18], [26].

### 3.3. Full access, infinite source, multi-queue model $M/M/n$

In this case the general delay system of Fig. 1 is considered with  $g > 1$  input groups or classes of calls with arrival rates  $\lambda_j, j = 1, 2, \dots, g$ .

#### 3.3.1. Stationary state probabilities

For a complete description of the system performance, various system state descriptions are defined which are also used for delay-loss systems (cf. Chapter 5):

- a)  $(x_1, \dots, x_g; c_1, \dots, c_z)$   $x_i$  number of group- $i$ -calls in service,  
 $c_j$  group index of that call waiting at the  $j$ -th position,
- b)  $(x_1, \dots, x_g; z_1, \dots, z_g)$   $x_i$  as a),  
 $z_j$  number of group- $j$ -calls waiting,
- c)  $(x; c_1, \dots, c_z)$   $x$  total number of calls in service without regard to their origin,  
 $c_j$  as a),
- d)  $(x; z_1, \dots, z_g)$   $x$  as c),  
 $z_j$  as b),

- e)  $(x; z)$   $x$  as c),  
 $z$  total number of wait-  
ing calls.

Because of full accessibility, the quantities  $c_j, z_j$ , and  $z$  are zero in all cases where  $x = x_1 + \dots + x_g < n$ .

For any of the disciplines FIFO, RANDOM, or LIFO with respect to all waiting calls it holds:

a)  $p(x_1, \dots, x_g; c_1, \dots, c_z) =$   

$$= p_0 \prod_{i=1}^g \left( \frac{A_i^{x_i}}{x_i!} \right) \prod_{j=1}^z \varrho_{c_j}, \quad (27)$$

b)  $p(x_1, \dots, x_g; z_1, \dots, z_g) =$   

$$= p_0 \prod_{i=1}^g \left( \frac{A_i^{x_i}}{x_i!} \right) z! \prod_{j=1}^g \left( \frac{\varrho_j^{z_j}}{z_j!} \right), \quad (28)$$

e)  $p(x; c_1, \dots, c_z) = \begin{cases} p_0 \frac{A^x}{x!}, & x < n, \\ p_0 \frac{A^n}{n!} \prod_{j=1}^z \varrho_{c_j}, & x = n, \end{cases} \quad (29)$

d)  $p(x; z_1, \dots, z_g) = \begin{cases} p_0 \frac{A^x}{x!}, & x < n, \\ p_0 \frac{A^n}{n!} \prod_{j=1}^g \left( \frac{\varrho_j^{z_j}}{z_j!} \right), & x = n, \end{cases} \quad (30)$

e)  $p(x; z) = \begin{cases} p_0 \frac{A^x}{x!}, & x < n, \\ p_0 \frac{A^n}{n!} \varrho^z, & x = n, \end{cases} \quad (31)$

where  $A_i = \lambda_i/\varepsilon, A = A_1 + \dots + A_g, \varrho_i = A_i/n,$   
 $\varrho = A/n.$

The proof of eq. (27) is carried out by insertion into the equilibrium state equation and stating consistency for all equations regarding the underlying queue discipline. Eqs. (28) to (31) can be found either directly by the same technique or by combining equally probable microstates to macrostates. Eq. (27) has been proved in case of FIFO along with queuing network theorems [1]. In case of RANDOM eq. (30) was derived directly [15].

Additionally, we state that there are  $g + 1$  partial equilibriums within the very microstate space. The most important one is that where each microstate is already in equilibrium with all its lower neighbour states. This property has already been observed in connection with multi-queue delay-loss systems, cf. [15], [16], [18], revealing some interesting aspects as, e.g., with respect to

- proof-technique,
- recursive calculation of the state probabilities,
- approximations.

Especially the last aspect was very successful for approximate solutions of multidimensional state spaces yielding excellent accuracy, cf. [30], [16], [18], and Chapter 5.

### 3.3.2. Characteristic values

Any of the distributions eqs. (27) to (30) yields the following results (index  $j$  refers to calls of group

number  $j$ , quantities without index refer to Section 3.1):

$$Y_j = A_j, \quad (32)$$

$$\Omega_j = p(n; 0) \frac{\varrho_j}{(1 - \varrho)^2} = \frac{\varrho_j}{\varrho} \Omega, \quad (33)$$

$$W_j = p(n; 0) \frac{1}{1 - \varrho} = W, \quad (34)$$

$$w_j = p(n; 0) \frac{h}{n(1 - \varrho)^2} = w, \quad (35a)$$

$$t_{Wj} = \frac{h}{n(1 - \varrho)} = t_W, \quad (35b)$$

$$W_j(> t) = W(> t), \quad j = 1, 2, \dots, g. \quad (36)$$

### 3.4. Full access, finite source, multi-queue model $M/M/n$

Now the general delay system of Fig. 1 is considered with  $g > 1$  input groups (queues), each having an individual number of sources  $q_j$  and idle source arrival rate  $\alpha_j, j = 1, 2, \dots, g$ . For short, only explicit solutions of state probabilities are given. The state description follows Section 3.3.

For any of the disciplines FIFO, RANDOM, or LIFO with respect to all waiting calls it holds:

a)  $p(x_1, \dots, x_g; c_1, \dots, c_z) = \quad (37)$

$$= p_0 \prod_{i=1}^g \left( \frac{\beta_i^{x_i}}{x_i!} \right) \prod_{j=1}^z \left( \frac{\beta_{c_j}}{n} \right) \prod_{k=1}^g \left( \prod_{v=0}^{x_k+z_k-1} (q_k - v) \right),$$

b)  $p(x_1, \dots, x_g; z_1, \dots, z_g) = \quad (38)$

$$= p_0 \prod_{i=1}^g \left( \frac{\beta_i^{x_i}}{x_i!} \right) z! \cdot \prod_{j=1}^g \frac{(\beta_j/n)^{z_j}}{z_j!} \prod_{k=1}^g \left( \prod_{v=0}^{x_k+z_k-1} (q_k - v) \right),$$

where  $\beta_i = \alpha_i/\varepsilon$ . The equations can be proved as outlined in Section 3.3.1.

The marginal (macro) state probabilities analogously to eqs. (29) and (30) are somewhat difficult to give in explicit form. Under a slight approximate assumption, however, an explicit solution can be derived in case of RANDOM and  $\alpha_i = \alpha, i = 1, 2, \dots, g$ :

$p(x; z_1, \dots, z_g) = \quad (39)$

$$= \begin{cases} p_0 \frac{\beta^x}{x!} \prod_{i=1}^{x-1} (Q - i), & x < n \\ p_0 \frac{\beta^n}{n!} \prod_{i=0}^{n-1} (Q - i) z! \cdot \prod_{j=1}^g \frac{(\beta/n)^{z_j}}{z_j!} \cdot \prod_{k=1}^g \left( \prod_{v=0}^{z_k-1} \left( q_k \left[ 1 - \frac{n}{Q} \right] - v \right) \right), & x = n \end{cases}$$

where  $\beta = \alpha/\varepsilon, Q = q_1 + q_2 + \dots + q_g$ . To prove this formula, we assume for  $x = n$  that the  $n$  calls being in service had been generated uniformly from the various source groups, i.e. we assume  $x_j \approx n \cdot q_j/Q, j = 1, 2, \dots, g$ . For  $g = 1$  this formula coincides with eq. (18).

### 3.5. Limited access, infinite source model $M/M/n(k)$

Now, the  $n$  servers are assumed to be graded with limited accessibility  $k$ . Both, ideal and real gradings are considered.

3.5.1. Stationary state probabilities

In the general case, there does not exist an explicit closed form solution. Exact results can only be obtained by iterative solution of equilibrium equations of microstate probabilities [16], [18], [19]. Here we only refer to approximate solutions for the symmetrical case  $\lambda_j = \lambda/g, j = 1, 2, \dots, g$ .

Defining the state  $(x, z)$  analogously to Section 3.3.1, the state space according to Fig. 4 is obtained.

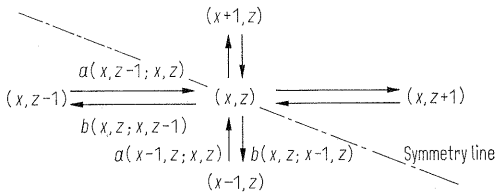


Fig. 4. State space and transitions of the  $M/M/n(k)$  delay system with limited access.

In case of an infinite number of sources the transition rates are:

$$a(x-1, z; x, z) = \lambda u(x), \quad x = 1, 2, \dots, n, \quad z \geq 0, \quad (40a)$$

$$a(x, z-1; x, z) = \lambda c(x), \quad x = k, k+1, \dots, n, \quad z \geq 1, \quad (40b)$$

$$b(x, z; x-1, z) + b(x, z; x, z-1) = x\varepsilon, \quad x = 1, 2, \dots, n \quad (40c)$$

where  $c(x)$  the blocking probability and  $u(x) = 1 - c(x)$  the passage probability of the grading according to

$$c(x) = \begin{cases} 0, & x = 0, 1, \dots, k-1, \\ \binom{x}{k} / \binom{n}{k}, & x = k, k+1, \dots, n \quad (\text{ideal gradings}), \\ \binom{x}{k^*} / \binom{n}{k^*}, & x = k, k+1, \dots, n \quad (\text{real gradings}). \end{cases} \quad (41)$$

In case of real gradings,  $c(x)$  is calculated using a reduced accessibility  $k^* = k^*(n, k, \rho, M, \text{interqueue disc.})$ , e.g. in case of standard gradings [19]

$$k^* \approx k - \frac{n^2 - k^2}{n^2} \left( \frac{k}{5} \rho + \frac{k-3}{2} \rho^{k/4} + a \frac{k}{5} \rho^4 \right) \frac{1}{M-1} \quad (42)$$

with  $a = 0$  or  $a = 1$  for interqueue discipline FIFO or RANDOM, respectively.

Under the special assumption of partial balance of state  $(x, z)$  with its lower neighbour states ("half-symmetry") the equilibrium state equation reads in the general case:

$$x\varepsilon p(x, z) = a(x-1, z; x, z) p(x-1, z) + a(x, z-1; x, z) p(x, z-1). \quad (43)$$

By insertion of eqs. (40a, b) and summation over all  $z$  the marginal state probabilities  $p(x)$  are obtained by recursion:

$$p(x) = p_0 A^x \prod_{i=0}^{x-1} u(i) / \prod_{i=1}^x [i - A c(i)], \quad x = 0, 1, \dots, n. \quad (44)$$

3.5.2. Characteristic values

$$\Omega = \sum_{x=k}^n \sum_{z=0}^{\infty} z p(x, z) = \sum_{x=k}^n p(x) \sum_{i=k}^x \frac{A c(i)}{i - A c(i)}, \quad (45)$$

$$W = \sum_{x=k}^n \sum_{z=0}^{\infty} p(x, z) c(x) = \sum_{x=k}^n p(x) c(x), \quad (46)$$

$$w = \frac{\Omega}{A} h, \quad t_W = \frac{w}{W}. \quad (47a, b)$$

The df of waiting times can be approximated by a gamma df [19] or even better by a Weibull df analogously to eqs. (15) and (16) using the following variation coefficient

$$c_W^2 \approx \begin{cases} \frac{8}{4 + \left(1 - \frac{k}{n}\right) \rho^3} - 1 & \text{for F/F,} \\ \frac{4}{2 - \rho^{n/k} (1 - M^{-2/3})} - 1 & \text{for R/F} \end{cases} \quad (48)$$

where F/F or R/F are short notations for interqueue/queue discipline FIFO (F) or RANDOM (R), respectively.

Eqs. (43) to (47) were derived by M. Thierer [30], [31] for ideal gradings ("Interconnection Delay Formula IDF"). Extensions with respect to real gradings and the df of waiting times were reported in [19].

3.6. Limited access, finite source model  $M/M/n(k)$

In this section the solution technique from Section 3.5 is extended to the finite source model with  $q_j = Q/g$  sources per grading group and  $\alpha_j = \alpha, j = 1, 2, \dots, g$ .

3.6.1. Stationary state probabilities

With the state definition  $(x, z), x = 0, 1, \dots, n, z = 0, 1, \dots, z_{\max}(x)$ , the same state diagram of Fig. 4 holds in the general case with

$$a(x-1, z; x, z) = [Q u(x-1) - (x-1)(1-d(x-1))] \alpha, \quad x = 1, 2, \dots, n, \quad z = 0, 1, \dots, z_{\max}(x), \quad (49a)$$

$$a(x, z-1; x, z) = [Q c(x) - x d(x) - z + 1] \alpha, \quad x = k, k+1, \dots, n, \quad z = 1, 2, \dots, z_{\max}(x), \quad (49b)$$

$$b(x, z; x-1, z) + b(x, z; x, z-1) = x\varepsilon, \quad x = 1, 2, \dots, n, \quad (49c)$$

where  $c(x)$  according to eq. (41), and  $d(x)$  that proportion of calls in service, originated by sources of momentary blocked grading groups. It was found [14] that  $d(x)$  can be approximated by

$$d(x) = g k c(x) / x M = n c(x) / x, \quad x = k, k+1, \dots, n. \quad (50)$$

Another complication is caused by the finiteness of the state space. Here, an upper number of waiting calls in state  $x$  is introduced by rounding up the expected number of sources being not in service within the blocked grading groups:

$$z_{\max}(x) = \begin{cases} 0, & x = 0, 1, \dots, k-1, \\ -\text{ENTIER}(-Qc(x) + xd(x)), & x = k, k+1, \dots, n. \end{cases} \quad (51)$$

The general equilibrium state equation reads analogously to eq. (43) with eqs. (49), (50), and (51). The two-dimensional state probabilities  $p(x, z)$  are calculated numerically by recursion. From  $p(x, z)$  the state probabilities  $\pi(x, z)$  seen by an arriving call are determined analogously to eq. (19):

$$\begin{aligned} \pi(x, 0) &= (Q - x)p(x, 0)/D, \quad x = 0, 1, \dots, k-1, \\ \pi(x, z) &= (Q - x - z)p(x, z)/D, \quad x = k, k+1, \dots, n, \\ &\quad z = 0, 1, \dots, z_{\max}(x) - 1, \end{aligned} \quad (52)$$

$$\pi(x, z_{\max}(x)) = [Qu(x) - x(1 - d(x))] \times p(x, z_{\max}(x))/D,$$

with

$$D = \sum_{i=0}^{k-1} (Q - i)p(i, 0) + \sum_{i=k}^n \left\{ \sum_{j=0}^{z_{\max}(i)-1} (Q - i - j)p(i, j) + [Qu(i) - i(1 - d(i))]p(i, z_{\max}(i)) \right\}.$$

### 3.6.2. Characteristic values

Based on  $p(x, z)$  and  $\pi(x, z)$  the following values are derived:

$$Y = A = \sum_{x=1}^n \sum_{z=0}^{z_{\max}(x)} x p(x, z), \quad (53)$$

$$\Omega = \sum_{x=k}^n \sum_{z=0}^{z_{\max}(x)} z p(x, z) = Q - A - \frac{A}{\beta}, \quad (54)$$

$$W = \sum_{x=k}^n \sum_{z=0}^{z_{\max}(x)-1} \pi(x, z) \frac{Qc(x) - xd(x) - z}{Q - x - z}, \quad (55)$$

$$w = \frac{\Omega}{A} h, \quad t_W = \frac{w}{W}. \quad (56a, b)$$

## 4. Delay Systems with Constant Holding Times

In this chapter single or multi-queue delay systems are considered having an infinite number of traffic sources with total arrival rate  $\lambda$  and constant holding time  $h$ .

### 4.1. Full access, infinite source model $M/D/n$

#### 4.1.1. Stationary state probabilities

For the exact analysis we follow the concept of describing the system state at time epochs  $t_0$ ,  $t_0 + h$ ,  $t_0 + 2h$ , ... introduced by C. D. Crommelin [8]. This special view allows the description by an imbedded Markov chain. For conformity the system states at that time epochs will again be indicated by  $(x, z)$ .

In the stationary state the following equilibrium equations hold:

$$p(x, 0) = a_n q_x + \sum_{i=1}^x p(n, i) q_{x-i}, \quad x < n, \quad (57a)$$

$$p(n, z) = a_n q_{n+z} + \sum_{i=1}^{n+z} p(n, i) q_{n+z-i}, \quad x = n, z \geq 0, \quad (57b)$$

$$\text{where } a_x = \sum_{i=0}^x p(i, 0), \quad q_i = \frac{A^i}{i!} e^{-A}. \quad (57c)$$

To solve eqs. (57a, b) usually the generating function of the stationary state probabilities  $p(x, z)$  is defined. The solution for the state probabilities involves  $n$  roots of a transcendental equation.

For numerical reasons a direct method is used to solve eqs. (57a, b) by iteration starting from an initial state distribution  $(M/M/n)$ . An upper value  $z_{\max}$  is determined such that  $|\Omega(z_{\max}) - \Omega|/\Omega < 10^{-8}$ . To keep the computing time as low as possible, a number of computational tricks had to be applied. By this method, systems with up to 250 servers and offered traffic per server up to 0.95 can be calculated relatively fast.

#### 4.1.2. Characteristic values

From the stationary state probabilities the characteristic mean values  $\Omega$ ,  $W$ ,  $w$ , and  $t_W$  can be evaluated by application of definitions given in eqs. (7) to (9).

The df of waiting times is explicitly known in case of FIFO with respect to all calls, cf. [8]:

$$W(>t) = 1 - \sum_{\mu=0}^r \sum_{\nu=0}^{n-1} a_{\nu} \frac{[-A(\mu + \tau)]^{(r-\mu+1)n-1-\nu}}{[(r-\mu+1)n-1-\nu]!} e^{A(\mu+\tau)}, \quad (58)$$

$$\text{where } t = (r + \tau)h, \quad 0 \leq \tau \leq 1.$$

The numerical evaluation, however, runs into complication for large delays. This problem can be overcome by programming with double-precision and by exponential continuation of  $W(>t)$  for large  $t$ . The second moment can be determined using a general relationship between the  $k$ -th factorial moment of the number of calls in the system and the  $k$ -th ordinary moment of the waiting time for queuing models  $M/G/n$  [10]. Using this theorem, we arrive at

$$c_{W, \text{FIFO}}^2 = \frac{WE[Z(Z-1)]}{\Omega^2} - 1, \quad (59)$$

where  $E[Z(Z-1)]$  is easily obtained from the numerically explicit state probabilities. In case of  $n=1$  the known result  $c_{W, \text{FIFO}}^2 = (2A+1)/3$  is obtained [29].

In case of RANDOM with respect to all waiting calls, an exact solution is known only for  $n=1$ , cf. P. J. Burke [6]. In the multi-server case  $n > 1$ , we approximate the second moment according to a heuristic similarity theorem [9], [21] using the results of the second moment of waiting time for queuing systems  $M/M/n$  FIFO,  $M/M/n$  RANDOM, and  $M/D/n$  FIFO. By that

$$c_{W, \text{RANDOM}}^2 \approx (2c_{W, \text{FIFO}}^2 + \rho)/(2 - \rho), \quad (60)$$

where  $c_{W, \text{FIFO}}^2$  according to eq. (59). This formula yields in case of  $n=1$  the exact result [29]. With  $t_W$  and  $c_{W, \text{RANDOM}}$  an approximate expression for the df of waiting times is obtained using the Weibull df according to eqs. (15), (16).



4.2. Limited access, infinite source model  $M/D/n(k)$

For this model M. Thierer [32] derived the stationary state equations based on the blocking probability  $c(x)$  for ideal gradings and a two-dimensional state  $(x, z)$  as in Section 3.5. Starting from that solution, the method has been extended to real gradings and the df of waiting times.

4.2.1. Stationary state probabilities

According to [32] the equations of state are as follows:

$$p(x, 0) = M(x, 0) \sum_{j=0}^x q_{x-j} \sum_{i=0}^n p(i, j), \quad x = 0, 1, \dots, n, \quad (61a)$$

$$p(x, z) = M(x, z) \sum_{j=0}^{x+z} q_{x+z-j} \sum_{i=0}^n p(i, j), \quad (61b)$$

$$x = k, \dots, n, \quad z \geq 0,$$

where  $(61c)$

$$M(x, z) = M(x-1, z)u(x-1) + M(x, z-1)c(x),$$

with  $q_i$  according to eq. (57c) and  $c(x)$  according to eq. (41) for ideal as well as real gradings ( $k^*$  according to eq. (42)).

For the numerical evaluation, an iteration method is used as explained in Section 4.1.1 with initial values from the system  $M/M/n(k)$  and an upper value  $z_{\max}$ . Using a number of computational methods for the reduction of computer storage and computation time, systems with up to 200 servers and with an offered traffic per server up to 0.95 have been calculated economically fast. The method was used for the calculation of tables [20].

4.2.2. Characteristic values

The characteristic values  $\Omega$ ,  $W$ ,  $w$ , and  $t_W$  are calculated from the state probabilities  $p(x, z)$  according to the definitions given in eqs. (45) to (47).

For the df of waiting times only the case of RANDOM interqueue and FIFO queue discipline has been considered (R/F). For the variation coefficient of waiting times a heuristic formula was derived from systematic simulation studies:

$$c_{W, R/F}^2 \approx \frac{3}{1.8 - \rho^{n/k}(1 - M^{-2/3})} - 1 \quad (62)$$

for  $n > k$ , and  $M \geq 2$ . With  $t_W$  and  $c_{W, R/F}$  the df of waiting times can be approximated by a Weibull df according to eqs. (15) and (16).

5. Delay-Loss Systems with Exponential Holding Times

In this chapter, combined delay and loss systems are considered, i.e. systems with a limited number of waiting places according to Table 1. For systems with general holding times, one server and finite waiting room ( $M/G/1-s$ ) there were also exact and approximate solutions derived for the df of waiting times in case of FIFO and RANDOM queue discipline, cf. [20], [21], which are not included in this paper.

5.1. Full access, infinite source, single queue model  $M/M/n-s$

5.1.1. Stationary state probabilities

Using the same notations as in Section 3.1, the only difference in the calculation of state probabilities lies in the finiteness of the state space in Fig. 3 with respect to  $z_{\max} = s$ . The state probabilities are given by eq. (5a, b).

5.1.2. Characteristic values

Analogously to Section 3.1:

$$Y = \sum_{x=0}^{n-1} x p(x, 0) + n \sum_{z=0}^s p(n, z) = A(1-B), \quad (63)$$

$$\Omega = \sum_{z=0}^s z p(n, z) = p(n, 0) \rho \left[ \frac{1-\rho^s}{(1-\rho)^2} - \frac{s\rho^s}{1-\rho} \right], \quad (64)$$

$$W = \sum_{z=0}^{s-1} p(n, z) = p(n, 0) \frac{1-\rho^s}{1-\rho}, \quad (65)$$

$$B = p(n, s) = p(n, 0) \rho^s, \quad (66)$$

$$w = W \frac{h}{n} \left[ \frac{1}{1-\rho} - \frac{s\rho^s}{1-\rho^s} \right], \quad t_W = \frac{w}{W}. \quad (67a, b)$$

The df of waiting times reads in case of FIFO

$$W(>t) = \frac{W}{1-\rho^s} \left[ \sum_{i=0}^{s-1} \frac{(\lambda t)^i}{i!} - \rho^s \sum_{i=0}^{s-1} \frac{(n \epsilon t)^i}{i!} \right] e^{-n \epsilon t}. \quad (68)$$

These results go back to H. Störmer [27]. For the queue disciplines RANDOM and LIFO, exact results for the df of waiting times and higher order moments have also been derived, cf. [17], [18]. In the exact solution of the df of waiting times, however, eigenvalues of a system of differential equations are involved.

In case of RANDOM queue discipline, approximate methods have been used according to the concept of approximating the conditional df of waiting times by exponential sums, as explained in Section 3.2.2, or according to the approximation of the total df of waiting times by a Weibull df as explained in Section 3.1.2. The latter method has also been used for the calculation of tables [20].

5.2. Full access, finite source, single queue model  $M/M/n-s$

5.2.1. Stationary state probabilities

By the same arguments as used in Section 5.1, the solution (18a, b) holds for  $p(x, z)$  also in case of a finite number of sources. The arriving call state distribution  $\pi(x, z)$  is obtained analogously to eq. (19) by substituting the upper bound  $(q-n)$  by  $s$ .

5.2.2. Characteristic values

The characteristic values  $Y$ ,  $\Omega$ ,  $W$ ,  $w$ ,  $t_W$ , and  $W(>t)$  are obtained from eqs. (20) to (24) by substituting  $(q-n)$  by  $s$  in the upper summation bounds (in a delay-loss system it holds  $s < q-n$ ). The loss probability  $B$  is equal to  $\pi(n, s)$ . These results are included in the more general treatment of this model with priorities [4].

In case of RANDOM queue discipline, the same principles can be applied for exact and approximate calculations as explained in Section 3.2.

### 5.3. Full access, infinite source, multi-queue model $M/M/n-s$

In the multi-queue delay-loss system two types of storage models will be distinguished: firstly, when there is a common storage capacity  $S$  for all calls, and secondly, when there is a limited storage capacity  $s_j$  exclusively for calls of group or class  $j$ ,  $j = 1, 2, \dots, g$ .

#### 5.3.1. Stationary state probabilities

Using the identical state definitions as introduced in Section 3.3.1 we state:

For any of the disciplines FIFO, RANDOM, LIFO with respect to all waiting calls the state distributions according to eqs. (27) to (31) hold also in case of delay-loss systems with common storage capacity  $S$  for all classes of calls ( $z \leq S$ ).

Such a general theorem does not apply in case of individual storages per class. Nevertheless, it can be proved:

For any of the disciplines FIFO, RANDOM, LIFO with respect to all waiting calls the marginal state distributions according to eqs. (29), (30) hold in case of delay-loss systems with an individual storage capacity  $s_j$  for calls of class  $j$  ( $z_j \leq s_j$ ),  $j = 1, 2, \dots, g$ .

The proof of these theorems again is done by inserting the results into the equilibrium state equations and stating consistency. In case of RANDOM, eq. (30) has been proved directly [15].

For the practical case where  $s_j = s$  and  $A_j = A/g$ , a special solution has been derived which is included in an algorithm given in Section 5.5.

#### 5.3.2. Characteristic values

Dependent on the underlying storage model, the characteristic values can be defined straightforwardly from the state distribution eq. (30). The definitions will not be reproduced here.

### 5.4. Full access, finite source, multi-queue model $M/M/n-s$

As in Section 3.4, we will only concentrate on the state probabilities. Using the same state definitions as introduced in Section 3.3.1, we find for the two considered storage models:

For any of the disciplines FIFO, RANDOM, LIFO with respect to all waiting calls the state distributions according to eqs. (37), (38) also hold in case of delay-loss systems with common storage capacity  $S$  for all classes of calls ( $z \leq S$ ).

Under the same approximate assumption as in Section 3.4, eq. (39) holds also in case of delay-loss systems with individual storage capacity  $s_j$  for calls of class  $j$  ( $z_j \leq s_j$ ),  $j = 1, 2, \dots, g$ .

### 5.5. Limited access, infinite source model $M/M/n(k)-s$

The exact analysis of graded delay-loss systems has been carried out under general assumptions by

numerical means, cf. [16], [18]. Here, we only confine ourselves to symmetrical systems having  $s_j = s$  waiting places and equal arrival rates  $\lambda_j = \lambda/g$  per grading group,  $j = 1, 2, \dots, g$ .

#### 5.5.1. Stationary state probabilities

Contrary to the state descriptions above, we define a state  $(x; u_1, \dots, u_s)$ , where

$x$  total number of calls in service,

$u_j$  total number of calls waiting on the  $j$ -th waiting place,  $j = 1, 2, \dots, s$ .

This description by "waiting place rows" reduces the number of states considerably [16]. The state space and its transitions are shown for the general case in Fig. 5.

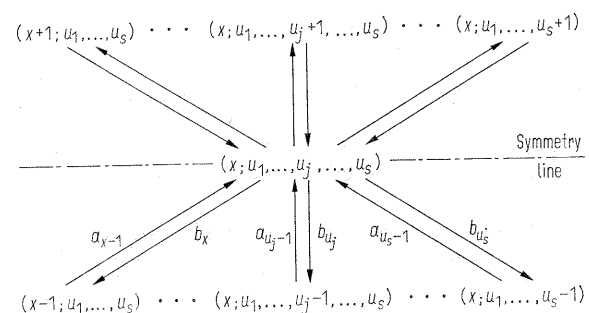


Fig. 5. State space and transitions of the  $M/M/n(k)-s$  delay-loss system with limited access.

With the general transition rates

$$a_{x-1} = \lambda[1 - c(x-1)], \quad (69a)$$

$$a_{u_1-1} = \lambda[g c(x) - (u_1 - 1)]/g, \quad (69b)$$

$$a_{u_j-1} = \lambda[u_{j-1} - (u_j - 1)]/g, \quad j = 2, 3, \dots, s, \quad (69c)$$

$$b_x + b_{u_1} + \dots + b_{u_s} = x \varepsilon \quad (69d)$$

and a partial equilibrium assumption (cf. Section 3.3.1), the following recursion formula can be derived:

$$x \varepsilon p(x; u_1, \dots, u_s) = a_{x-1} p(x-1; u_1, \dots, u_s) + \sum_{j=1}^s a_{u_j-1} p(x, \dots, u_j-1, \dots) \quad (70)$$

with

$$u_s \leq u_{s-1} \leq \dots \leq u_1 \leq -\text{ENTIER}(-g c(x)), \quad (71)$$

and  $c(x)$  according to eq. (41). For real gradings, the value  $k^*$  of pure delay systems [18] has to be modified by substituting  $\rho = A/n$  by  $Y/n$  and by inserting a reduction term  $s/(s+1)$ ; for standard gradings, e.g., we have

$$k^* \approx k - \frac{n^2 - k^2}{n^2} \left[ \frac{k}{5} \left( \frac{Y}{n} \right) + \frac{k-3}{2} \left( \frac{Y}{n} \right)^{k/4} + a \frac{k}{5} \left( \frac{Y}{n} \right)^4 \right] \frac{1}{M-1} \frac{s}{s+1}, \quad s \geq 1 \quad (72)$$

instead of eq. (42). Since  $Y$  is a result, iterations have to be performed [14]. Finally, we state that eq. (70) yields the exact results in case of full access.

5.5.2. Characteristic values

$$Y = \sum_{x=0}^n \sum_{u_1=0}^g \cdots \sum_{u_s=0}^g x p(x; u_1, \dots, u_s) = A(1 - B), \tag{73}$$

$$\Omega = \sum_{x=k}^n \sum_{u_1=0}^g \cdots \sum_{u_s=0}^g (u_1 + \cdots + u_s) p(x; u_1, \dots, u_s), \tag{74}$$

$$W = \sum_{x=k}^n \sum_{u_1=0}^g \cdots \sum_{u_s=0}^g \left[ c(x) - \frac{u_s}{g} \right] p(x; u_1, \dots, u_s), \tag{75}$$

$$B = \sum_{x=k}^n \sum_{u_1=0}^g \cdots \sum_{u_s=0}^g \frac{u_s}{g} p(x; u_1, \dots, u_s), \tag{76}$$

$$w = \frac{\Omega}{A} h, \quad t_W = \frac{w}{W}. \tag{77 a, b}$$

5.6. Limited access, finite source model  $M/M/n(k) - s$

5.6.1. Stationary state probabilities

The method of Section 5.5 has been extended to a finite number of  $q_j = q = Q/g$  sources per grading group with  $\alpha_j = \alpha, j = 1, 2, \dots, g$ . The transition rates for the state space according to Fig. 5 are:

$$a_{x-1} = [Qu(x-1) - (x-1)(1-d(x-1))] \alpha, \tag{78 a}$$

$$a_{u_1-1} = [gc(x) - (u_1-1)] \left( q - \frac{xd(x)}{gc(x)} \right) \alpha, \tag{78 b}$$

$$a_{u_j-1} = [u_{j-1} - (u_j-1)] \left( q - (j-1) - \frac{xd(x)}{gc(x)} \right) \alpha, \tag{78 c}$$

$j = 2, 3, \dots, s,$

$$b_x + b_{u_1} + \cdots + b_{u_s} = x\varepsilon \tag{78 d}$$

with  $c(x)$  according to Section 5.5 and  $d(x)$  according to eq. (50). Analogously to Section 5.5.1, the state probabilities are obtained from eq. (70) with transition rates according to eqs. (78 a-d). From  $p(x; u_1, \dots, u_s)$  the arriving call state distribution  $\pi(x; u_1, \dots, u_s)$  is found:

$$\pi(x; u_1, \dots, u_s) = \frac{(Q - x - u_1 - \cdots - u_s) p(x; u_1, \dots, u_s)}{\sum_{i=0}^n \sum_{j_1=0}^g \cdots \sum_{j_s=0}^g (Q - i - j_1 - \cdots - j_s) p(i; j_1, \dots, j_s)} \tag{79}$$

5.6.2. Characteristic values

The carried traffic  $Y$ , the mean number of waiting calls  $\Omega$ , and the mean waiting times  $t_W$  and  $w$  follow directly according to definitions given in eqs. (73), (74), and (77 a, b). For the probabilities of delay  $W$  and of loss  $B$  it holds

$$W = \sum_{x=k}^n \sum_{u_1=0}^g \cdots \sum_{u_s=0}^g \frac{Qc(x) - xd(x) - u_1 - \cdots - u_s - u_s \left[ q - \frac{xd(x)}{gc(x)} - s \right]}{Q - x - u_1 - \cdots - u_s} \pi(x; u_1, \dots, u_s), \tag{80}$$

$$B = \sum_{x=k}^n \sum_{u_1=0}^g \cdots \sum_{u_s=0}^g u_s \left[ q - \frac{xd(x)}{gc(x)} - s \right] \pi(x; u_1, \dots, u_s). \tag{81}$$

Finally, we note for reasons of completeness that the corresponding solutions for graded loss systems are reported in [5], [23], [24], [11], and [2] both for ideal and real gradings.

(to be continued)

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# Graded Delay Systems with Infinite or Finite Source Traffic and Exponential or Constant Holding Time\*

## Part II: Numerical Results and Validation

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In Part I of the paper (cf. AEÜ 32 [1978], 8–18), theoretical results have been reported for various single-stage delay systems. Part II presents a number of numerical results for the major traffic characteristics together with simulation results for validation of approximate analysis methods.

**Einstufige Wartesysteme mit vollkommener oder unvollkommener Erreichbarkeit, unendlicher oder endlicher Quellenzahl und exponentiellen oder konstanten Belegungsdauern**

**Teil II: Numerische Ergebnisse und Genauigkeitsnachweis**

Im Teil I des Beitrags (siehe AEÜ 32 [1978], 8–18) wurde über theoretische Ergebnisse für verschiedenartige einstufige Wartesysteme berichtet. Im Teil II werden für die wichtigsten Verkehrsgrößen numerische Ergebnisse angeführt und zur Überprüfung approximativer Rechenverfahren mit Simulationsergebnissen verglichen.

### 6. Numerical Results and Validation

Part I of this paper dealt with the theoretical analysis methods for various delay systems with full and limited accessibility, infinite and finite source traffic and an unlimited or limited number of waiting places. Parts of those methods are based on approximations to make the solution feasible for evaluation and application.

In this chapter, some calculated results are given to show the influence of various system and traffic parameters on the main characteristic values, and to check the accuracy of approximate results by comparison with simulations.

#### 6.1. Full access, single queue models $M/M/n$

The mean values of the single queue delay system  $M/M/n$  with full access and finite or infinite number of sources are known exactly. In Fig. 6 the df's of waiting times  $W(>\tau)/W$  are given for systems with  $n = 10$  servers,  $q = 20$  and  $q \rightarrow \infty$  traffic sources, and service according to the queue disciplines FIFO (dashed curves) and RANDOM (solid curves), respectively. The results for FIFO hold exactly whereas the results for RANDOM are approximations according to Sections 3.1.2 and 3.2.2. The approximate results are in good accordance with simulations.

#### 6.2. Limited access models $M/M/n(k)$

Fig. 7 presents the probability of delay  $W$  and the relative mean waiting time of waiting calls  $\tau_W = t_W/h$  dependent on the offered traffic per server  $A/n$  for a limited access delay system with  $n = 30$  and  $k = 10$ . The solid curves and the simulation results hold for a standard grading with  $M = 3^{1/3}$ ,  $q = 15$  sources per group, and R/F

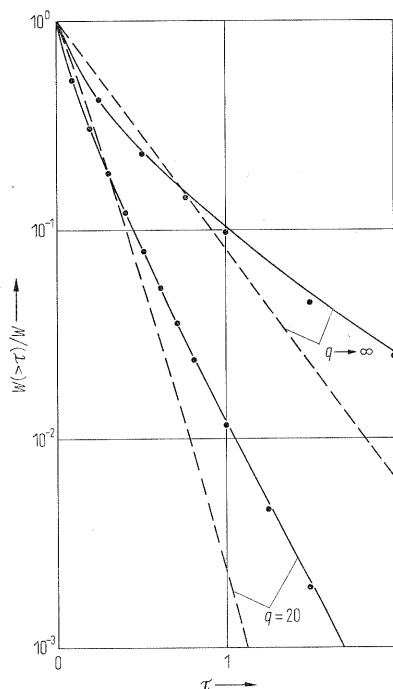
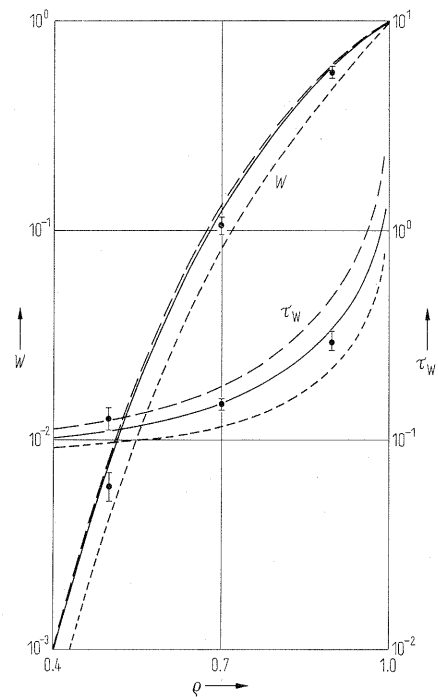
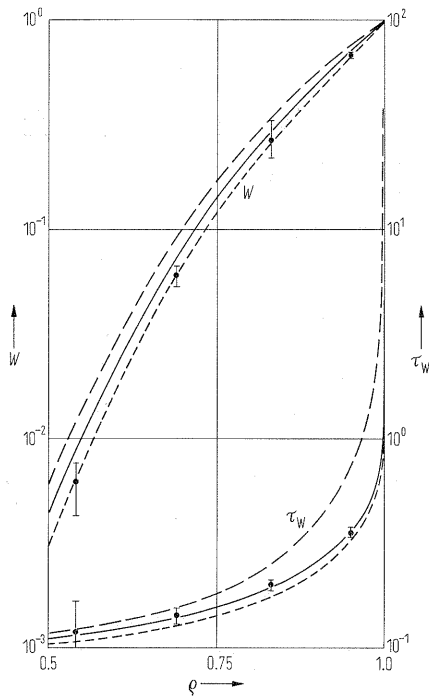


Fig. 6. Distribution function of waiting times of waiting calls  $W(>\tau)/W$  versus normalized time  $\tau = t/h$ . Full access delay systems  $M/M/n$ ;  $n = k = 10$ ,  $g = 1$ ; number of sources:  $q = 20, \infty$ ; queue disciplines: FIFO (---), RANDOM (—); • simulation; offered traffic per server  $\rho = 0.75$ .

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Figs. 7 and 8. Probability of delay  $W$  and relative mean waiting time of waiting calls  $\tau_w$  versus offered traffic per server  $\rho = A/n$ .

Limited access delay systems  $M/M/n(k)$ ;  $n = 30$ ,  $k = 10$ ,  $M = 3^{1/3}$  (standard grading); number of sources per group:  $q = 15, \infty$ ; interqueue/queue discipline: R/F;  
 — standard grading,  $q = 15/\text{group}$ ,  
 - - - standard grading,  $q \rightarrow \infty/\text{group}$ ,  
 ···· ideal grading,  $Q = 150$ ;  
 ● simulation with 95% confidence intervals.

Limited access delay systems  $M/D/n(k)$  and  $M/M/n(k)$ ;  $n = 30$ ,  $k = 10$ ,  $M = 2$  (standard grading); number of sources per group:  $q \rightarrow \infty$ ; interqueue/queue discipline: R/F;  
 —  $M/D/n(k)$ , standard grading,  
 - - -  $M/M/n(k)$ , standard grading,  
 ····  $M/D/n(k)$ , ideal grading;  
 ● simulation with 95% confidence intervals.

discipline. For comparison, the corresponding results are given for this grading for  $q \rightarrow \infty$  and R/F discipline (dashed curves) as well as in case of an ideal grading with a total of  $Q = 150$  sources (dotted curves).

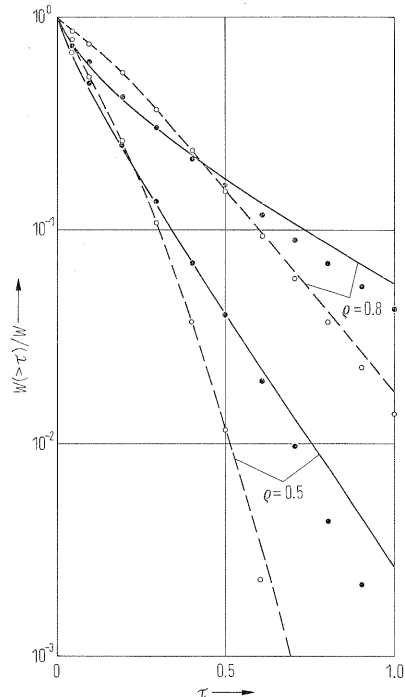
6.3. Limited access models  $M/D/n(k)$

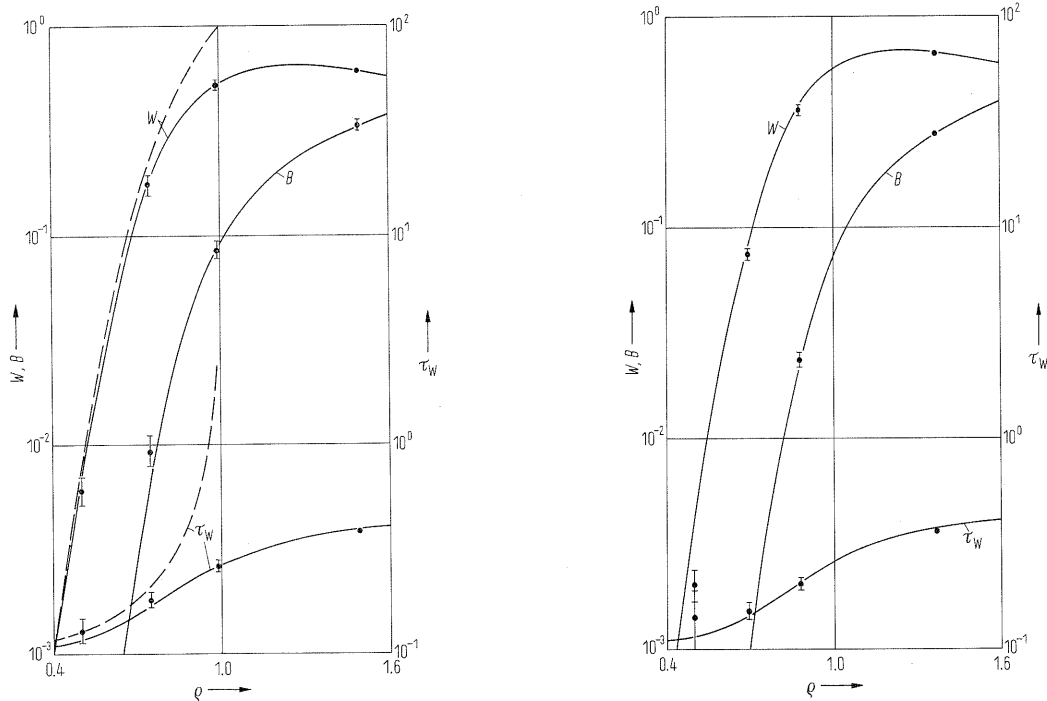
In Fig. 8 the characteristic values  $W$  and  $\tau_w$  are drawn versus  $A/n$  for gradings with  $n = 30$ ,  $k = 10$ . The solid curves and simulations hold in case of a standard grading with  $M = 2$ , R/F discipline, and constant holding time. For comparison, the corresponding results are given for this grading with exponential holding time (dashed curves) and for an ideal grading with constant holding time (dotted curves).

Note, that the holding time characteristic influences mainly  $\tau_w$  whereas  $W$  remains nearly unchanged (real gradings).

Fig. 9. Distribution function of waiting times of waiting calls  $W(> \tau)/W$  versus normalized time  $\tau = t/h$ . Full access delay systems  $M/D/n$ ;  $n = k = 10$ ,  $g = 1$ ; number of sources:  $q \rightarrow \infty$ ; queue disciplines: FIFO (---), RANDOM (—); ● ○ simulation.

$\tau_w$		$q = 0.5$	$q = 0.8$
FIFO	calculation	0.1431	0.2855
	simulation	0.1423	0.2808
RANDOM	calculation	0.1431	0.2855
	simulation	0.1482	0.2813





Figs. 10 and 11. Probability of delay  $W$ , probability of loss  $B$ , and relative mean waiting time of waiting calls  $\tau_w$  versus  $\rho = A/n$ .

Limited access delay-loss system  $M/M/n(k) - s$ ;  $n = 30$ ,  $k = 10$ ,  $M = 2$  (standard grading);  
 number of waiting places/group:  $s = 3, \infty$ ;  
 number of sources per group:  $q \rightarrow \infty$ ;  
 interqueue/queue discipline: R/F;  
 —  $M/M/n(k) - s$ , - - -  $M/M/n(k)$ ,  
 • simulation with 95% confidence intervals.

Limited access delay-loss system  $M/M/n(k) - s$ ;  $n = 30$ ,  $k = 10$ ,  $M = 2$  (standard grading);  
 number of waiting places/group:  $s = 3$ ;  
 number of sources per group:  $q = 20$ ;  
 interqueue/queue discipline: R/F;  
 —  $M/M/n(k) - s$ , finite sources;  
 • simulation with 95% confidence intervals.

#### 6.4. Full access, single queue model $M/D/n$

In Fig. 9 the df's of waiting times are shown for the delay system  $M/D/n$  with  $n = 10$  for FIFO (dashed curves) and RANDOM (solid curves) queue disciplines, respectively. In case of RANDOM, a two-moment approximation has been used according to Section 4.1.2. Compared with simulations, eq. (60) generally yields a slightly too high variation coefficient which is also reflected in Fig. 9.

#### 6.5. Limited access models $M/M/n(k) - s$

Figs. 10 and 11 give results for the limited access delay-loss system with standard grading having  $n = 30$ ,  $k = 10$ ,  $M = 2$ , and  $s = 3$  waiting places per grading group. In Fig. 10 an infinite source traffic is assumed. For comparison, the corresponding pure delay system results are added (dashed curves). Fig. 11 presents the corresponding results for the finite source model with a total of  $Q = 120$  traffic sources. For both, the infinite and finite source delay-loss system with limited access, the calculated results are sufficiently accurate for applications.

### 7. Conclusion

In this paper, a number of single-stage delay systems has been reviewed in a systematical way.

Starting from known results, which are either briefly repeated or referred to, various extensions have been presented with respect to multi-queues (classes), finite number of traffic sources, limited accessibility, and different queue disciplines. The results were partly used for tables on delay systems and form basic modules for the analysis of multi-stage connecting arrays (link systems) with waiting. The extensions in case of multi-class delay systems are also applicable to Markovian queuing networks with respect to queue disciplines RANDOM and LIFO. In case of multi-class delay-loss systems two subcases of limited storage capacity are distinguished: common storage for all classes or individual storage per class. The general product solutions of pure delay systems can partially be extended to those delay-loss systems. All presented solutions are adequate for a computational evaluation; approximate solutions were checked by simulations yielding sufficient accuracy for practical applications (dimensioning).

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