

# Calculation of Fully Available Groups and Gradings for Mixed Pure Chance Traffic

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## 1. Introduction

In general, traffic problems are treated for uniform pure chance traffic of Type 1 (PCT1) or uniform pure chance traffic of Type 2 (PCT2; definitions cf. Section 2). The well known results [2, 3] give information about the total grade of service.

In the following it is shown how to calculate all characteristic traffic values, if at the same time PCT1 and PCT2 – so called mixed pure chance traffic – are offered to a fully available group or grading. These results are important, if we want to specify the grade of service for special incoming trunk groups.

Exact results are obtained for all switching networks with full availability. An approximate solution has been found for gradings. The evaluation of a large number of artificial traffic tests shows the good accordance between calculated and simulated values.

## 2. Definitions of Traffic and Abbreviations

### PCT1 (Pure Chance Traffic of Type 1)

An infinite number of sources produces the offered traffic with the mean value  $A_1$ . The total call rate  $\lambda_1$  is constant and independent of the number of busy sources.

### PCT2 (Pure Chance Traffic of Type 2)

A finite number of sources produces the offered traffic  $A_2$ . Each free source has the constant call rate  $\alpha_2$ .

In both cases the sources are supposed to be independent from each other. Idle sources start calls at random. This implies a negative exponential distribution of idle times of each source. The distribution of holding times is also assumed in both cases to be negative exponential with the mean value  $t_m$  (termination rate:  $\epsilon_1 = \epsilon_2 = 1/t_m$ ).

Therefore, the offered traffic is given by the following equations:

$$\text{PCT 1: } A_1 = \lambda_1 t_m = \lambda_1 / \epsilon,$$

$$\text{PCT 2: } A_2 = (q - Y_2) \alpha_2 / \epsilon,$$

where  $Y_2$  is the share of traffic carried on the trunk group, referring to PCT 2.

$k$	Availability (accessibility) per selector multiple
$n$	Number of trunks (lines) in the outgoing group (route)
$q$	Number of traffic sources
$\lambda_1$	Call rate in case of PCT1
$\alpha_2$	Call rate per idle source in case of PCT2
$\epsilon$	Termination rate
$\beta_2$	Abbreviation for $\alpha_2/\epsilon$
$A$	Offered traffic
$Y$	Carried traffic
$E$	Time congestion probabilities
$B$	Call congestion probabilities
$x$	Instantaneous number of existing occupations (calls) in a group (state $\{x\}$ )

$p(x)$	Probabilities of a state $\{x\}$
$u(x)$	Passage probability in the state $\{x\}$ of a trunk group
$c(x)$	Blocking probability in the state $\{x\}$ of a trunk group; $c(x) = 1 - u(x)$
$f_2(x)$	Partition probability. Probability, that an established call is originated by the finite number of PCT2-sources
Indices 1	for values, referring to PCT1
Indices 2	for values, referring to PCT2
$v = \frac{A_1}{A_1 + A_2}$	traffic factor (relative share of PCT1)

## 3. Equilibrium Equations – Calculation of the Probabilities of State

Taking into account the special character of PCT1 and PCT2 we can find directly the equations for statistical equilibrium:

$$A_1 \cdot p(x) \cdot u(x) + [q_2 - x \cdot f_2(x)] \cdot \beta_2 \cdot p(x) \cdot u(x) = [x + 1] \cdot p(x + 1). \quad (1)$$

Because both PCT1 and PCT2 have to be in statistical equilibrium we can share Eqn. (1) into two parts (the general proof may be obtained by writing down the bidimensional equations of state  $p(x_1, x_2)$  and some simple transformations).

Hence we can write for PCT1

$$A_1 \cdot p(x) \cdot u(x) = [x + 1] \cdot [1 - f_2(x + 1)] \cdot p(x + 1), \quad (2)$$

and for PCT2

$$[q_2 - x \cdot f_2(x)] \cdot \beta_2 \cdot p(x) \cdot u(x) = [x + 1] \cdot f_2(x + 1) \cdot p(x + 1). \quad (3)$$

Substituting  $(x-1)$  for  $x$  in Eqn. (2) and combining this new Eqn. (2) with Eqn. (3) we can find an equation containing no longer the partition probabilities:

$$A_1 \cdot \beta_2 \cdot u(x) \cdot p(x - 1) + [q_2 \cdot \beta_2 \cdot u(x) - x \cdot u(x) \cdot \beta_2 + A_1 u(x)] \cdot p(x) = [x + 1] \cdot p(x + 1). \quad (4)$$

This linear homogeneous difference equation of second order can be solved by introducing continued fraction. The solution is given by

$$p(x) = \frac{1}{x!} \prod_{i=0}^{x-1} K_i \quad (5)$$

$$1 + \sum_{r=1}^n \frac{1}{r!} \prod_{i=0}^{r-1} K_i$$

with

$$K_i = [q_2 \beta_2 - i \beta_2 + A_1] u(i) + \frac{i A_1 - \beta_2 u(i - 1) \cdot u(i)}{K_{i-1}} \quad (0 \leq i \leq n)$$

$$K_0 = [q_2 \beta_2 + A_1].$$

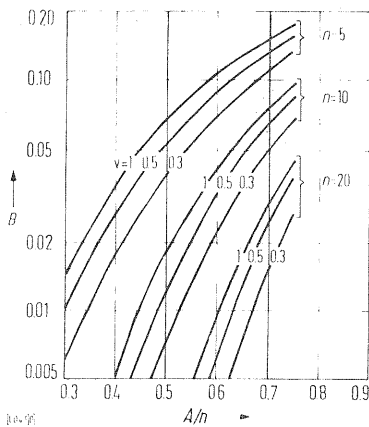


Fig. 1

Fig. 1. Total call congestion  $B$  as a function of the relative share  $v$  of PCT 1 and the offered traffic per trunk  $A/n$ .

Fig. 2

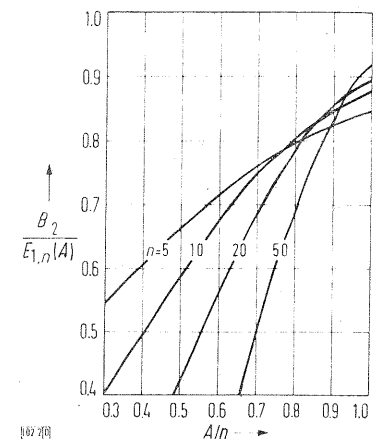


Fig. 2. Relative call congestion for the share of PCT2-traffic;  $q_2=n$ ,  $v=0.5$ .

#### 4. Time and Call Congestion

Time congestion is the same for both PCT1 and PCT2:

$$E_1 = E_2 = \sum_{x=k}^n p(x) c(x). \quad (6)$$

Call congestion for PCT1:

$$B_1 = E_1 = \sum_{x=k}^n p(x) \cdot c(x). \quad (7)$$

Call congestion for PCT2:

$$B_2 = \frac{q_2 \cdot E_2 - \sum_{x=k}^n x \cdot f(x) \cdot c(x) \cdot p(x)}{q_2 - Y_2} \quad (8)$$

with

$$Y_2 = \sum_{x=0}^n x \cdot f(x) \cdot p(x).$$

Total call congestion:

$$B = v B_1 + (1 - v) \cdot B_2. \quad (9)$$

#### 5. Blocking Probability $c(x)$

Obviously, we can find for fully available groups  $c(x)=0$  for  $(0 \leq x < n)$  and  $c(n)=1$ .

For gradings two types of blocking probabilities have been regarded. First, the well known formula for ideal Erlang-gradings [5]

$$c(x) = \frac{\binom{x}{k}}{\binom{n}{k}} \quad (10)$$

and second a well fitted approximation of Kirsch [4], which modifies  $c$  dependent on the special type of the considered grading - the blocking probability of Erlang by an empirical approach

$$c(x) = \frac{\binom{x}{k^*}}{\binom{n}{k^*}} \quad (11)$$

with

$$k^* = k - c_1 \cdot [k - 5] \cdot \frac{n - k}{n} \left[ \frac{Y}{n} - c_2 \sqrt[4]{k - 6} \right], k \geq 6.$$

$c_1 = 1.30$  for simplified standard gradings of the German post office,

$c_1 = 2.40$  for O'Dell Gradings,

$c_2 = 0.32$  for both simplified standard and O'Dell gradings.

#### 6. Discussion of Results

Making use of the formulas, calculated in the previous sections, it is possible to give a lot of special informations about the grade of service. Only some results are presented in the following figures.

Fig. 1 shows the dependence of the total call congestion from the traffic factor  $v$  for fully available groups [ $v = 1$  corresponds to a hundred per cent share of PCT1, i. e. the Erlang Loss Formula  $E_{1,n}(A)$ ]. Detailed investigations have shown that the difference between the exact loss formula (Enqs. 7, 8 and 9) and  $E_{1,n}(A)$  is negligible for practical applications, if the number  $q_2$  of PCT2-traffic sources is greater than about five times the number  $n$  of trunks.

An important question is: what is the difference between the call congestion for the share of PCT1 (or PCT2, respectively) and the call congestion, if there would be only PCT1 ( $v=1$ , Erlang Loss Tables). Fig. 2 shows some results for PCT2-traffic.

As mentioned before, all results obtained for networks with full availability are calculated exactly. For gradings, however, approximate solutions are presented. Therefore, more than 200 simulation results for the total call congestion  $B$  and the partial values  $B_1$  and  $B_2$  have been compared with the above presented approximations. The comparison shows, that, assuming the blocking probability  $c(x)$  according to Erlang's formula, the results often tend to undervalue and sometimes to overestimate slightly reality. Making use of Kirsch's formula, we can find well approximated values in general and only sometimes a slight overestimate. Figures 3 and 4 show some typical results (simulation results with 95 % confidence intervals within the circles).

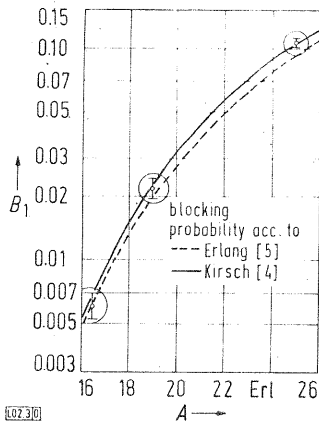


Fig. 3

Fig. 3. Call congestion  $B_1$  for a simplified standard grading of the German PO ( $n=30, k=10, g=6, q_2=18, v=0.6$ ; cf. Sections 2 and 6).

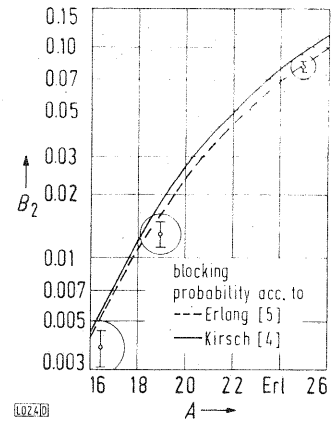


Fig. 4

Fig. 4. Call congestion  $B_2$  for a simplified standard grading of the German PO ( $n=30, k=10, g=6, q_2=18, v=0.6$ ; cf. Sections 2 and 6).

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