

THE PHASE CONCEPT:
APPROXIMATION OF MEASURED DATA AND PERFORMANCE ANALYSIS

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Distribution functions (d.f.) for interarrival and service times are described by means of a simplified Cox model with a uniform service rate for all phases but different branching probabilities.

We first present an efficient algorithm to determine a phase-type d.f. which fits actual measured data with a prescribed accuracy. Investigating queueing models with that type of d.f. it is shown how to reduce the complexity of state spaces.

The paper contains the following sections:

1. Introduction
2. Modeling with measured data
3. Analysis by means of the special phase concept
4. Numerical results
5. Conclusion and outlook

1. INTRODUCTION

One of the most famous methods for the exact or approximate analysis of queueing systems is the phase concept (method of stages), introduced by Erlang [1]. It is based on the idea of representing generally distributed interarrival or service times by sums of convolutions of exponentially distributed random variables, the so-called "phases".

Due to the memoryless property of the exponential distribution, the stochastic processes, described at the phase level, are of the Markovian type, so that the well-known method of analysis can be applied.

Typical and successfully treated examples are queueing systems with Erlangian or hyperexponentially distributed interarrival and service times, respectively [2].

1.1 GENERAL COX MODEL

Cox [3] generalized the concept in showing that any distribution function (d.f.) having a rational Laplace transform can be represented by a simple sequence of exponentially distributed phases, as shown in Figure 1.

Figure 1 should be interpreted in the following way:

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With probability α_0 the service time is equal to zero*. With probability $\beta_0 = 1 - \alpha_0$ a request enters the first exponentially distributed phase with mean μ_1^{-1} . After leaving this phase it enters the second phase (mean μ_2^{-1}) with probability β_1 or leaves the server with probability $\alpha_1 = 1 - \beta_1$, etc.

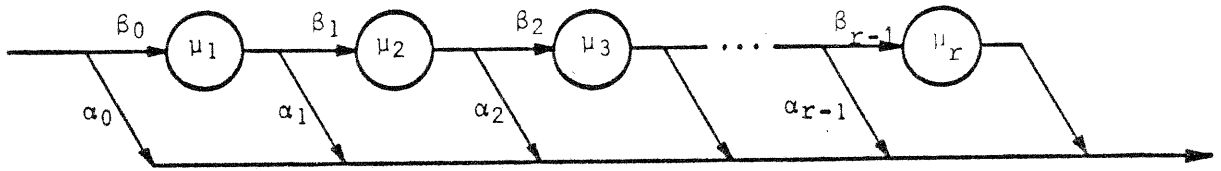


Fig.1. Representation of d.f. with rational Laplace transform by exponentially distributed phases, according to Cox ("general Cox model")

The Laplace transform of the probability density function, generated in this way, is given by

$$\Phi(s) = \alpha_0 + \sum_{i=1}^r \beta_0 \beta_1 \dots \beta_{i-1} \alpha_i \prod_{j=1}^i \frac{\mu_j}{s + \mu_j} \quad (1)$$

Since any d.f. may be approximated arbitrarily closely by rational functions the phase concept can in principle be used in very general cases.

However, two main problems arise in applying this general concept:

- 1) No effective method is known to determine all parameters of the d.f. (1), if actual measured values are given. (The approach which is mostly proposed for approximation [2,3], namely Prony's method [4], has several severe disadvantages [5].)
- 2) The complexity of state spaces and transitions increases remarkably in most cases. Hence, the evaluation of the state probabilities may become an extremely difficult numerical problem.

In the following it is shown how to overcome both problems for a wide range of application.

1.2 COX MODEL WITH UNIFORM SERVICE RATE

The key to the solution is to simplify the general Cox model by assuming a uniform, real service rate for all phases:

$$\mu_1 = \mu_2 = \dots = \mu_r = \mu \quad (2)$$

This leads to the following d.f.**:

$$F(t) = \sum_{i=1}^r q_i (1 - e^{-\mu t}) \sum_{j=0}^{i-1} \frac{(\mu t)^j}{j!} \quad t \geq 0 \quad (3)$$

$$q_i = \beta_1 \beta_2 \dots \beta_{i-1} \alpha_i \quad i \in \{1, 2, \dots, r\}$$

Despite the simplification the class of functions in (3) still possesses the following important property:

* We refer to service times here. Of course, Fig.1 may stand for interarrival times, too.

** For the sake of a simplified description, only d.f.s $F(t)$ with $F(0) = 0$ are considered, i.e. $\beta_0 = 1$.

For any probability d.f. $F(t)$ with $F(t) = 0$ for $t < 0$, there exists a series of functions (3) which converges weakly to $F(t)$. [6].

This means that measured data still may be approximated with any required accuracy.

We investigated the phase concept with this type of model. Although the number of phases may increase, we found the advantage of this approach to be twofold:

- 1) An efficient algorithm can be formulated to determine a d.f. of the type given in (3) which fits measured data with a prescribed accuracy (c.f. chapter 2).
- 2) The complexity of state spaces can often be drastically reduced, so that frequently a large amount of computing time and memory is saved in evaluating characteristic traffic values (c.f. chapter 3).

2. MODELING WITH MEASURED DATA

In this chapter we present an algorithm that allows to determine an approximating d.f. of the phase-type with uniform service rates, if a set of measured data is given.

2.1 APPROXIMATION TASK

The problem to be solved may be stated as follows:

Be given n discrete values $F(t_v)$ of a d.f. $F(t)$ at points t_v , $v \in \{1, 2, \dots, n\}$, as well as the values for the first moment M_1 and the second moment M_2 of $F(t)$ *.

Find a d.f. $A(t)$ so that the following four conditions are fulfilled (δ_v^u, δ_v^o are prescribed values):

$$A(t_v) \leq F(t_v) + \delta_v^o \quad (4)$$

$$A(t_v) \geq F(t_v) - \delta_v^u \quad (5)$$

$$\int_0^{\infty} t \, dA = M_1 \quad v \in \{1, \dots, n\} \quad (6)$$

$$\int_0^{\infty} t^2 \, dA = M_2 \quad (7)$$

2.2 MOTIVATION

Compared to standard approximation criteria our approach possesses two special features:

- 1) Mixed conditions: Experience shows that even if a (standard) approximation fits well the form of a given d.f. the moments of both functions may differ significantly. The use of such approximations is unsatisfactory (e.g. in a pure delay system even the value for the server utilization might be incorrect).

Therefore we use a mixture of conditions which prescribe the form of the d.f. within a certain tolerance scheme as well as the values of the moments of low order.

- 2) Limited deviations: After the approximating d.f. has been found, a goodness-of-fit test may be performed to determine whether the measured data can be considered as being obtained from the hypothesized distribution. The most suitable procedure, the Kolmogorov-Smirnov test [7], uses as a criterion for rejection the maximum deviation between the hypothesized d.f. and the sample d.f.

By an appropriate choice of the maximum deviations δ_v^u, δ_v^o in (4) and (5), we may achieve that the approximation is not rejected when this test is applied.

*The proposed method is not restricted to two moments, as described here for reasons of convenience.

2.3 APPROXIMATION PROCEDURE

For the reasons described above the class of functions defined in (3) is taken for approximation.

To fulfill the relations (4) to (7) the approximation task can be formulated as the following optimization problem:

Find the minimum number r of phases and the values of the parameters q_1, \dots, q_r and μ under the following restrictions:

$$\sum_{i=1}^r q_i \left(1 - e^{-\mu t_v} \sum_{j=0}^{i-1} \frac{(\mu t_v)^j}{j!} \right) \leq F(t_v) + \delta_v^o \quad (8)$$

$v \in \{1, \dots, n\}$

$$\sum_{i=1}^r q_i \left(1 - e^{-\mu t_v} \sum_{j=0}^{i-1} \frac{(\mu t_v)^j}{j!} \right) \geq F(t_v) - \delta_v^u \quad (9)$$

$$\mu^{-1} \sum_{i=1}^r i q_i = M_1 \quad (10)$$

$$\mu^{-2} \sum_{i=1}^r i(i+1) q_i = M_2 \quad (11)$$

$$\sum_{i=1}^r q_i = 1 \quad (12)$$

$$q_1, \dots, q_r \geq 0 \quad (13)$$

$$\mu \geq 0 \quad (14)$$

Our algorithm for the solution of this problem is described in the following.

- To fulfill equations (10) and (11) the range of the parameter r has a lower boundary, dependent on the values of M_1 and M_2 [8]:

$$r \geq \frac{1}{c^2} \quad \text{if } c^2 \leq 1 \quad (15)$$

$$r \geq 2c^2 + 2\sqrt{c^4 - 1} \quad \text{if } c^2 > 1 \quad (16)$$

$$c^2 = \frac{M_2 - M_1^2}{M_1^2}$$

- r is set to the lowest possible value and a solution is searched as described below. In case of non success search, r is incremented by one and again a solution is searched, etc.
- Searching values μ, q_1, \dots, q_r for a given r that fulfill conditions (8) to (14) is performed in the following way:

In each of the $2n+3$ relations (8) to (12) a nonnegative auxiliary variable h_k is introduced, and their sum is defined as a new variable z

$$z = \sum_{k=1}^{2n+3} h_k \quad (17)$$

The problem is to find a zero of z . Since z is nonnegative this problem is equivalent to finding out whether the minimum of z is zero or not. This is done in two separate parts:

- For a given μ we are faced with a linear optimization problem with variables q_1, \dots, q_r . This problem is solved using the well-known simplex algorithm [9]. The minimum value found is a (nonlinear) function of μ :

$$z_{\min} = z_{\min}(\mu) \tag{18}$$

- It remains to determine a minimum of z_{\min} ; this can be performed with the aid of known algorithms for the optimization of a nonlinear function of one variable [10]. The latter problem can be simplified, because boundaries for the location of a zero of z_{\min} can be derived, dependent on r , M_1 and M_2 . The interval I , which may contain a zero of z_{\min} , if any, is given by [8]:

$$I = \begin{cases} \left[\frac{1}{c^2 M_1}, \frac{1}{M_1} \sqrt{\frac{r(r+1)}{1+c^2}} \right] & \text{if } c^2 \leq 1 \\ \left[\frac{1}{M_1}, \frac{1}{M_1} \sqrt{\frac{r(r+1)}{1+c^2}} \right] & \text{if } c^2 > 1 \end{cases} \quad c^2 = \frac{M_2 - M_1^2}{M_1^2} \tag{19}$$

- As mentioned above, r is incremented by one, if the minimum of z is found to be greater than zero. This is repeated until a prescribed maximum value r_{\max} is reached.

2.4 EXAMPLES

Figures 2,3 show two examples for our method of approximation [8]. In both examples five values of the d.f. were prescribed with different deviations δ . In all cases the first and second moment were kept exactly at the prescribed values.

It is interesting to see how the approximation approaches the prescribed values when the allowed tolerance δ becomes smaller and smaller. At the same time the corresponding number of phases increases.

Another example, demonstrating the applicability of phase-type d.f.s is presented in chapter 4.

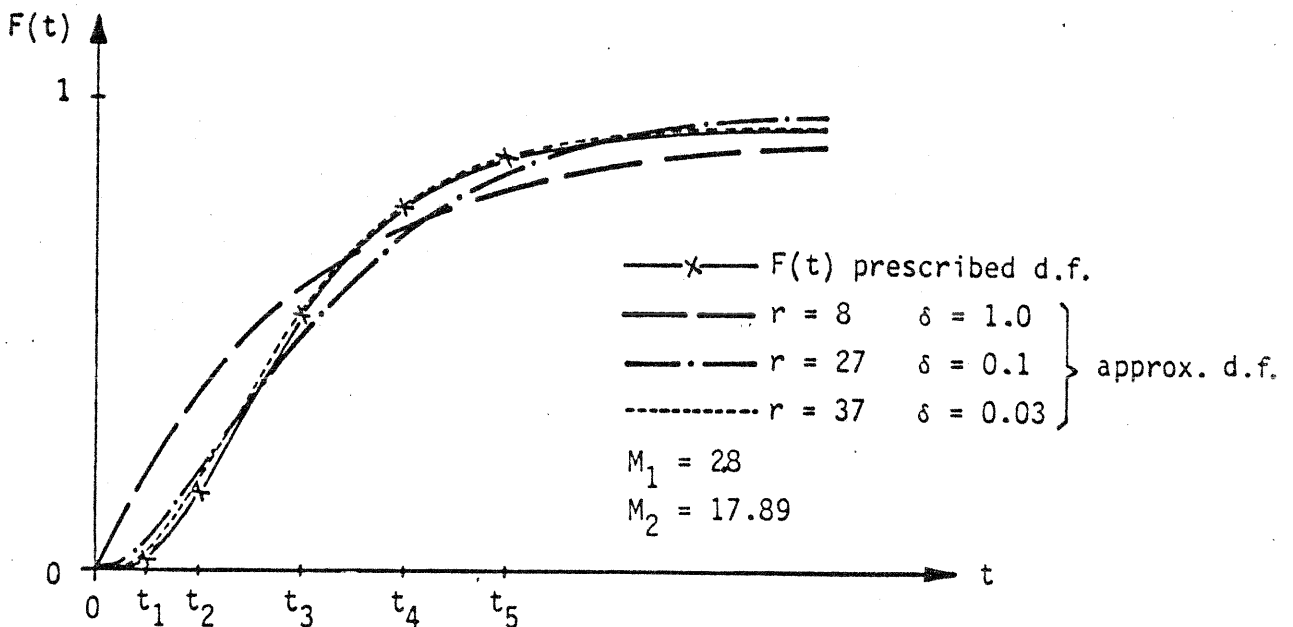


Fig.2. Approximation of a distribution function. The first two moments of the approximation are exactly equal to the prescribed values M_1 and M_2 , respectively. The form of the d.f. is kept within the prescribed tolerance scheme, defined by the maximum deviation δ (cf. text).

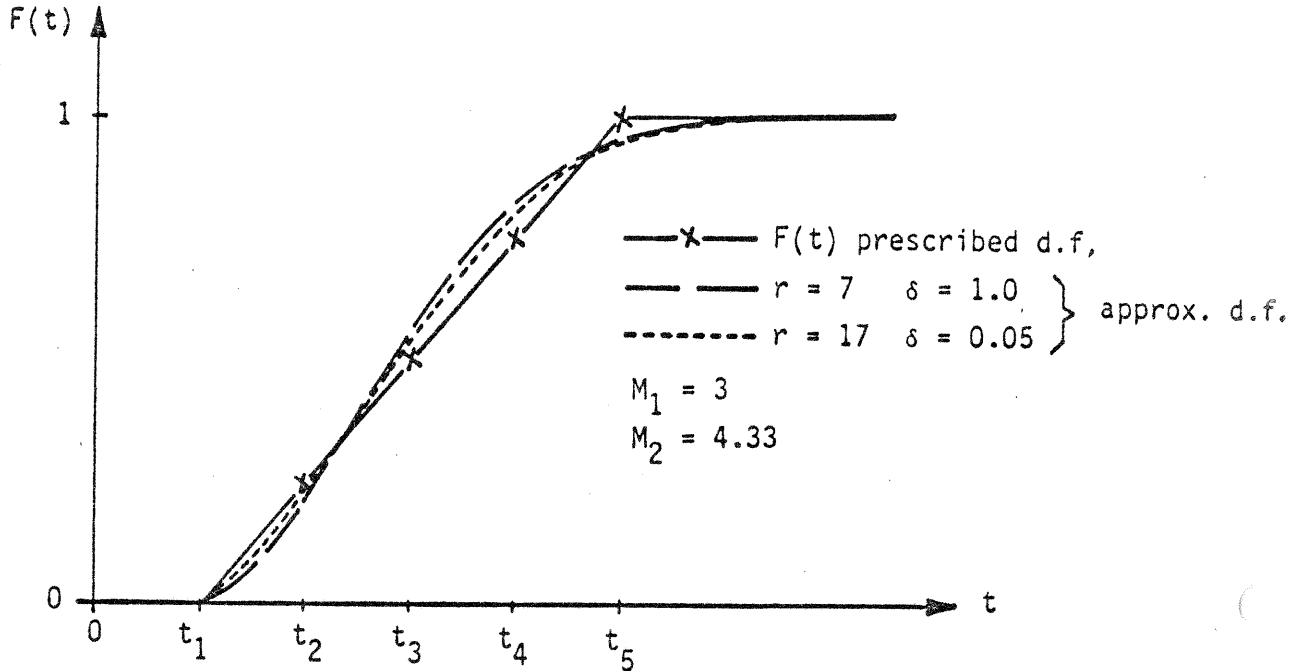


Fig.3. Second example for the approximation of a distribution function.
For comments see Fig.2.

3. ANALYSIS BY MEANS OF THE SPECIAL PHASE CONCEPT

This chapter describes how the phase concept with uniform service rates of all phases is effectively used in the analysis of queueing systems. After some general considerations two examples are considered.

3.1 GENERAL CONSIDERATIONS

3.1.1 STATE DESCRIPTION

If the general Cox model is used to describe interarrival or service times, then the state description of a queueing system usually includes:

- the number of requests in the system
- a variable describing the instantaneous phase of service for the request being processed
- a corresponding variable for the phase-type arrival process
- additional variables depending on the particular type of the system.

In case of an Erlangian distributed service time a well-known trick to reduce the complexity of the state description is to use a state variable defined as the total number of "unfinished" service phases (phases not yet completed by all requests in the system) [2,11]. A newly arriving request is equivalent to a bulk arrival of phases with fixed bulk size k , the order of the Erlangian distribution.

The concept of bulk arrival of phases can also be applied in case of the d.f. (3) but here with random bulk size i which occurs with probability q_i , $i \in \{1, \dots, r\}$ [6]. Although the knowledge of the actual number of requests in the system is lost in this case, the interesting performance values of most systems can be determined, as shown in the succeeding sections.

In describing the arrival process by means of a phase-type model there is no great difference regarding the complexity of the state space between the general Cox model and the simplified model with uniform service rates.

3.1.2 NUMERICAL EVALUATION

The "normal" way of solution for a system of equilibrium equations is to use generating functions. Usually, this leads to the problem of determining the roots of a polynomial equation, if a closed form solution for the generating function of the state probabilities can be found, at all. Since often numerical problems arise with this solution another approach is more favourable in many cases, namely a recursive technique [12].

3.2 EXAMPLE M/G/1

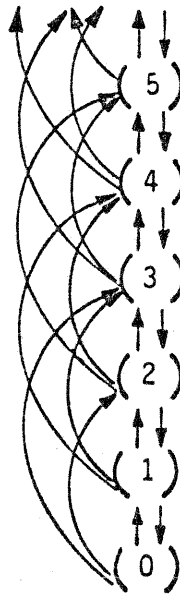
The usefulness of the special phase concept is demonstrated most evidently by analyzing the queuing system M/G/1.

3.2.1 STATE DESCRIPTION AND EQUILIBRIUM EQUATIONS

We assume that the d.f. of service times be represented by a phase-type function with uniform service rate μ and branching probabilities $q_i, i \in \{1, \dots, r\}$, according to equation (3).

The system state is described by means of a random variable X defined as the total number of service phases yet to be completed by all requests in the system. The corresponding state-transition diagram is shown in Fig. 4.

Fig.4. State-transition diagram for the queueing system M/G/1 (here $r=3$)



Herewith the equilibrium equations are as follows, if the mean arrival rate of the Poisson process is denoted by λ and for convenience q_i is set to zero, if $i > r$:

$$\begin{aligned}
 p(0) \lambda &= p(1) \mu \\
 p(x) (\lambda + \mu) &= \sum_{i=0}^{x-1} p(i) \lambda q_{x-i} + p(x+1) \mu \quad x > 0
 \end{aligned} \tag{20}$$

3.2.2 SOLUTION BY MEANS OF GENERATING FUNCTIONS

The generating function of the state probabilities is defined as:

$$F(z) = \sum_{x=0}^{\infty} z^x p(x) \tag{21}$$

The solution of equations (20) is derived in a straightforward manner by multiply-

ing the k-th equation by z^k and summing all equations. This leads to:

$$F(z) = \frac{\mu(z-1)}{(\lambda+\mu)z - \lambda zQ(z) - \mu} p(0) \quad (22)$$

$Q(z)$ is the generating function of the branching probabilities q_1 :

$$Q(z) = \sum_{i=0}^{\infty} z^i q_i = \sum_{i=0}^r z^i q_i \quad (23)$$

The probability $p(0)$ is given by:

$$p(0) = 1 - \frac{\lambda}{\mu} \sum_{i=1}^r i q_i \quad (24)$$

From equation (22) the moments of the distribution $p(x)$ can be derived; this leads for example to the Pollaczek-Khintchine formula of the mean waiting time in the system M/G/1.

If we are interested in the explicit values of the probabilities $p(x)$, the generating function $F(z)$ has to be retransformed. Then we are faced with the problem of determining the zeroes for the polynomial in the denominator. According to the degree r of $Q(z)$ this may become a difficult numerical problem.

3.2.3 SOLUTION BY MEANS OF A RECURSIVE EVALUATION

An alternative approach to the use of generating functions is to evaluate the state probabilities recursively. It follows immediately from equations (20):

$$\begin{aligned} p(1) &= \frac{\lambda}{\mu} p(0) \\ p(x+1) &= \frac{\lambda+\mu}{\mu} p(x) - \frac{\lambda}{\mu} \sum_{i=0}^{x-1} p(i) q_{x-i} \quad x > 0 \end{aligned} \quad (25)$$

In practice, the computation of the probabilities $p(x)$ is carried out for all x with $p(x)$ greater than a prescribed boundary.

3.2.4 CHARACTERISTIC PERFORMANCE VALUES

Once the state probabilities are known the interesting performance values can be determined (first-come-first-served is assumed):

- Moments of the waiting time d.f.:

$$E[T_w^m] = \frac{1}{\mu^m} \sum_{i=1}^{\infty} \frac{(i+m-1)!}{(i-1)!} p(i) \quad (26)$$

- Waiting time d.f.:

$$P(T_w \leq t) = \sum_{i=0}^{\infty} p(i) \left(1 - e^{-\mu t} \sum_{j=0}^{i-1} \frac{(\mu t)^j}{j!} \right) \quad t \geq 0 \quad (27)$$

Of course, there is no gain in evaluating the mean time in queue by means of equation (26), whereas in many cases the numerical evaluation of the waiting time d.f. is much easier and faster to carry out in the described way than by retransforming its well-known Laplace-transform, as usually done.

Numerical examples are given in chapter 4.

3.3 EXAMPLE $G/E_k/1,s$

This chapter demonstrates the application of the special phase concept to systems with a general input process. To our knowledge, there is no exact method available for the direct solution (via generating functions, etc.) which allows an efficient evaluation [14]. However, a recursive solution is possible and outlined in the following sections.

3.3.1 STATE DESCRIPTION AND EQUILIBRIUM EQUATIONS

It is assumed that the d.f. of interarrival times is represented by a phase-type d.f. with uniform rates. (Our way of solution is also applicable for nonuniform service rates; however, no method is available to fit measured data by that type of d.f., cf. chapter 1.1.) For the service process we assume the well-known Erlangian d.f.

The following abbreviations will be used:

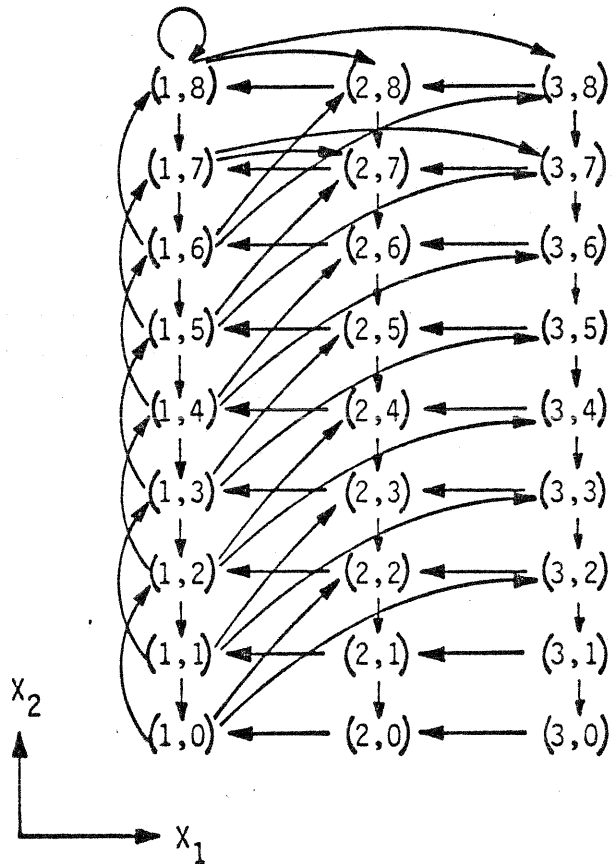
- λ : uniform rate for all phases in the interarrival time model
- q_i : branching probabilities of the interarrival time model, $i \in \{1, \dots, r\}$
- μ : uniform rate for all k phases in the Erlangian service time model
- s : number of waiting places.

The system state is described by means of a two-dimensional vector (X_1, X_2) , where

- X_1 : number of phases yet to be completed within the interarrival time model until arrival of the next request ($X_1 \in \{1, 2, \dots, r\}$)
- X_2 : total number of service phases yet to be completed by all requests in the system ($X_2 \in \{0, 1, \dots, x_{2max}\}$)

Figure 5 shows an example for the corresponding state-transition diagram.

Fig.5. State-transition diagram for the queueing system $G/E_k/1,s$ (here $r=3, k=2, s=3, x_{2max}=k(s+1)=8$)



The equilibrium equations for the state probabilities are as follows:

$$\underline{x_2=0:}$$

$$p(x_1, 0) \cdot \lambda = p(x_1, 1) \cdot \mu + p(x_1+1, 0) \cdot \lambda$$

$$\underline{1 \leq x_2 \leq x_{2\max} - k:}$$

$$p(x_1, x_2) (\lambda + \mu) = p(x_1, x_2+1) \mu + p(x_1+1, x_2) \cdot \lambda + p(1, x_2-k) \lambda q_{x_1} \quad (28)$$

$$\underline{x_2 > x_{2\max} - k:}$$

$$p(x_1, x_2) (\lambda + \mu) = p(x_1, x_2+1) \mu + p(x_1+1, x_2) \cdot \lambda + p(1, x_2-k) \cdot \lambda q_{x_1} + p(1, x_2) \cdot \lambda q_{x_1}$$

3.3.2 SOLUTION BY MEANS OF A RECURSIVE EVALUATION

To solve equations (28) a recursive technique is applied [12,13]. From equations (28) it follows immediately:

$$p(x_1, 1) = p(x_1, 0) \cdot \frac{\lambda}{\mu} - p(x_1+1, 0) \frac{\lambda}{\mu}$$

$$p(x_1, x_2+1) = p(x_1, x_2) \cdot \left(\frac{\lambda}{\mu} + 1\right) - p(x_1+1, x_2) \cdot \frac{\lambda}{\mu} - p(1, x_2-k) \cdot \frac{\lambda}{\mu} \cdot q_{x_1} \quad \text{if } 1 \leq x_2 \leq x_{2\max} - k \quad (29)$$

$$p(x_1, x_2+1) = p(x_1, x_2) \cdot \left(\frac{\lambda}{\mu} + 1\right) - p(x_1+1, x_2) \cdot \frac{\lambda}{\mu} - p(1, x_2-k) \cdot \frac{\lambda}{\mu} \cdot q_{x_1} - p(1, x_2) \cdot \frac{\lambda}{\mu} \cdot q_{x_1} \quad \text{if } x_{2\max} - k < x_2 \leq x_{2\max}$$

These equations show that all state probabilities $p(x_1, x_2)$ could be determined if the probabilities $p(x_1, 0)$, $x_1 \in \{1, \dots, r\}$, the so-called boundaries were known.

Therefore, the main steps of our recursive evaluation are [12,13]:

- Determine all remaining state probabilities as a function of the boundary values. This is possible when we introduce the substitution

$$p(x_1, x_2) = \sum_{\gamma=1}^r c_{x_1, x_2}^{\gamma} p(\gamma, 0) \quad (30)$$

in equations (29) and compare the coefficients of the probabilities $p(\gamma, 0)$ on both sides of the equations. The following expressions are obtained immediately ($\gamma \in \{1, \dots, r\}$):

$$\begin{aligned}
 C_{x_1,1}^{\gamma} &= C_{x_1,0}^{\gamma} \cdot \frac{\lambda}{\mu} - C_{x_1+1,0}^{\gamma} \cdot \frac{\lambda}{\mu} \\
 C_{x_1,x_2+1}^{\gamma} &= C_{x_1,x_2}^{\gamma} \cdot \left(\frac{\lambda}{\mu} + 1\right) - C_{x_1+1,x_2}^{\gamma} \cdot \frac{\lambda}{\mu} \\
 &\quad - C_{1,x_2-k}^{\gamma} \cdot \frac{\lambda}{\mu} \cdot q_{x_1} \quad \text{if } 1 \leq x_2 \leq x_{2\max} - k
 \end{aligned} \tag{31}$$

$$\begin{aligned}
 C_{x_1,x_2+1}^{\gamma} &= C_{x_1,x_2}^{\gamma} \cdot \left(\frac{\lambda}{\mu} + 1\right) - C_{x_1+1,x_2}^{\gamma} \cdot \frac{\lambda}{\mu} \\
 &\quad - C_{1,x_2-k}^{\gamma} \cdot \frac{\lambda}{\mu} \cdot q_{x_1} \\
 &\quad - C_{1,x_2}^{\gamma} \cdot \frac{\lambda}{\mu} \cdot q_{x_1} \quad \text{if } x_{2\max} - k < x_2 < x_{2\max}
 \end{aligned}$$

Since the coefficients

$$C_{x_1,0}^{\gamma} = \begin{cases} 1 & \text{if } \gamma = x_1 \\ 0 & \text{if } \gamma \neq x_1 \end{cases} \tag{32}$$

are known by definition (cf. equation (30)), all remaining coefficients can be determined recursively by equations (31).

- Solve a reduced system of only r equations for the unknown boundaries. This reduced system of equations is obtained by making substitution (30) to the remaining $(r-1)$ independent equations of (29) as well as to the normalizing condition:

$$\begin{aligned}
 0 &= \sum_{\gamma=1}^r C_{x_1,x_{2\max}}^{\gamma} \cdot p(\gamma,0) \cdot \left(\frac{\lambda}{\mu} + 1\right) \\
 &\quad - \sum_{\gamma=1}^r C_{x_1+1,x_{2\max}}^{\gamma} \cdot p(\gamma,0) \cdot \frac{\lambda}{\mu} \\
 &\quad - \sum_{\gamma=1}^r C_{1,x_{2\max}-k}^{\gamma} \cdot p(\gamma,0) \cdot \frac{\lambda}{\mu} \cdot q_{x_1} \\
 &\quad - \sum_{\gamma=1}^r C_{1,x_{2\max}}^{\gamma} \cdot p(\gamma,0) \cdot \frac{\lambda}{\mu} \cdot q_{x_1} \quad x_1 \in \{1, \dots, r-1\} \\
 1 &= \sum_{x_1=1}^r \sum_{x_2=0}^{x_{2\max}} \sum_{\gamma=1}^r C_{x_1,x_2}^{\gamma} \cdot p(\gamma,0)
 \end{aligned} \tag{33}$$

- Determine all interesting state probabilities by means of the (now known) boundaries and equations (29).

3.3.3 CHARACTERISTIC PERFORMANCE VALUES

Due to the non-Markovian input process, an important value is the probability $\pi(x_2)$, the probability that x_2 phases are in the system at the arrival instants. This conditional probability is given by:

$$\pi(x_2) = \frac{p(1, x_2)}{\sum_{x_2=0}^{x_{2max}} p(1, x_2)} = p(1, x_2) \sum_{i=1}^r i q_i \quad (34)$$

These probabilities allow to find the following expressions for the interesting performance values:

- Probability of waiting:

$$W = \sum_{x_2=1}^{x_{2max}} \pi(x_2) \quad (35)$$

- Moments of the waiting time d.f. (first-come-first-served):

$$E[T_W^m] = \mu^{-m} \sum_{i=1}^{x_{2max}} \frac{(i+m-1)!}{(i-1)!} \pi(i) \quad (36)$$

- Waiting time d.f. (first-come-first-served):

$$P(T_W \leq t) = \pi(0) + \sum_{i=x_{2max}-k+1}^{x_{2max}} \pi(i) + \sum_{i=1}^{x_{2max}-k} \pi(i) (1 - e^{-\mu t} \sum_{j=0}^{i-1} \frac{(\mu t)^j}{j!}) \quad t \geq 0 \quad (37)$$

Numerical examples are given in chapter 4.

4. NUMERICAL RESULTS

We presented in Figure 3 a special d.f. suitable for the modeling of access and transfer times for fixed head disks or drums. The corresponding results for the waiting time d.f. are shown in Figure 6.

Fig.6. Waiting time d.f. for a fixed head disk or drum model (M/G/1 ; service times acc. to Fig.3 A: offered traffic in Erlangs)

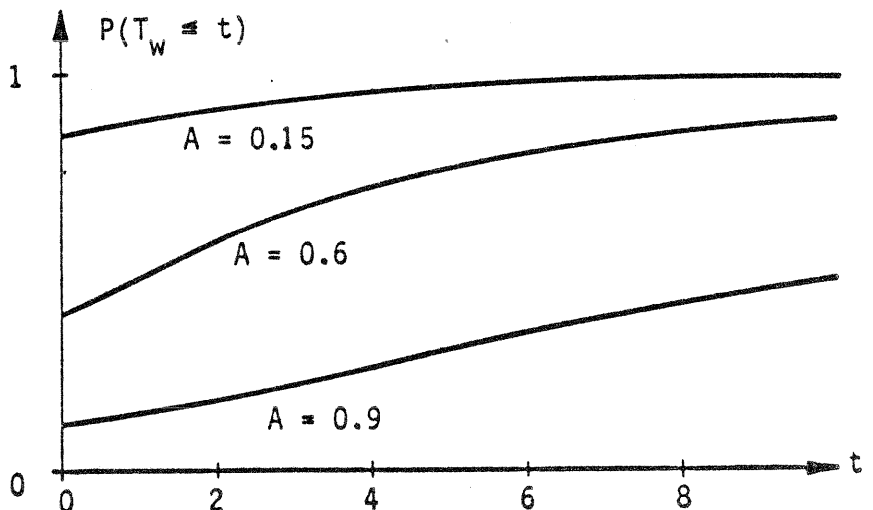


Figure 7 shows another example for a service time d.f. approximated by a phase-type d.f. with uniform service rates. Again, Figure 8 presents results for the corresponding waiting time d.f.

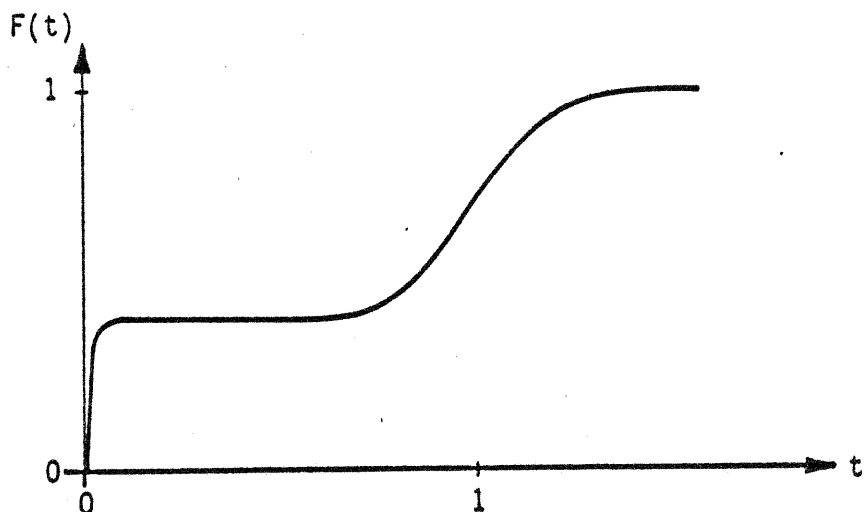


Fig.7. Third example for the approximation of a service time d.f. by a phase-type d.f. with uniform service rates.
 ($\mu=40$, $r=40$, $q_1=0.4$, $q_{40}=0.6$, $E[T]=0.61$, $\text{Var}[T]=0.24$)

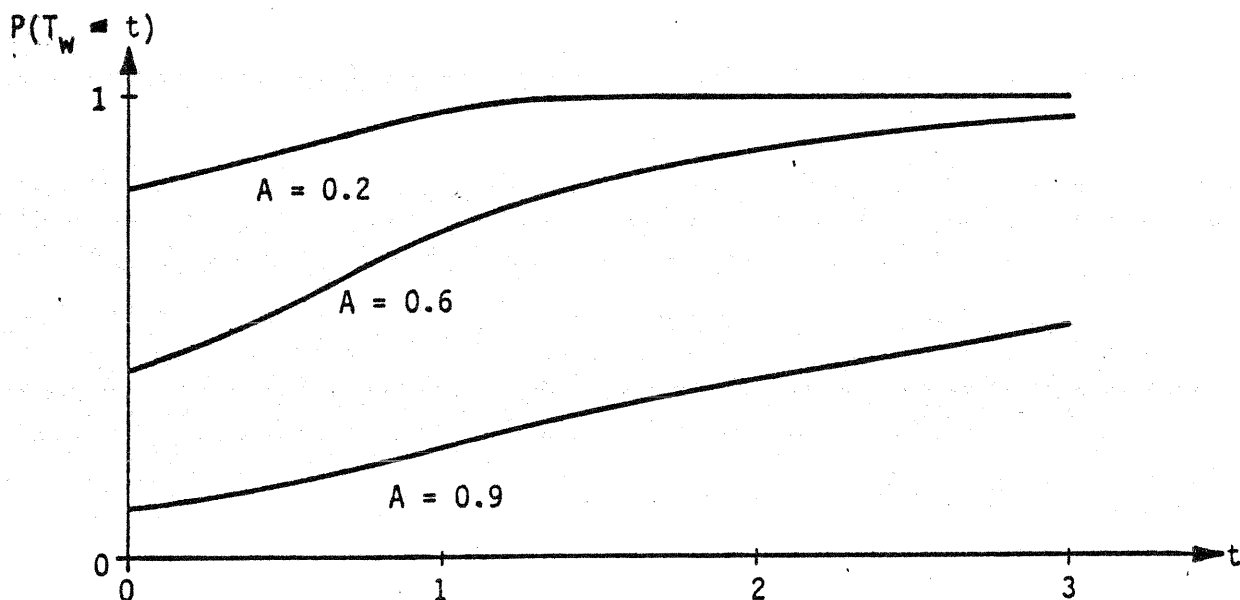


Fig.8. Waiting time d.f. for the above presented service time model.
 (M/G/1 ; A: offered traffic in Erlangs)

Finally, Figures 9 and 10 summarize results for state probabilities and the waiting time d.f. in case of a $G/E_k/1,s$ queueing system (the corresponding state transition diagram is sketched in Figure 5).

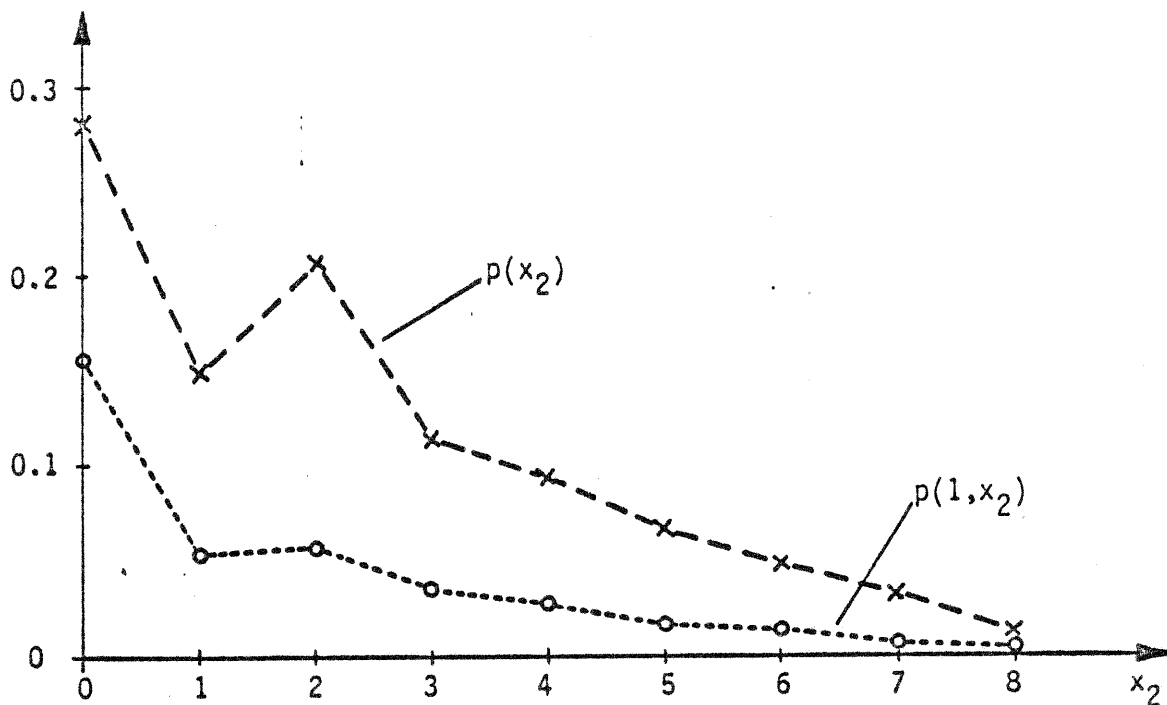


Fig.9. State probabilities for the queueing system $G/E_k/1, s$.
 (corresponding state-transition diagram cf. Fig.5 ;
 $r=3, k=2, s=3, x_{2max}=k(s+1)=8, \lambda=1, q_1=0.1, q_2=0.1,$
 $q_3=0.8, \mu=1$; mean arrival rate $\lambda_m=0.37$)

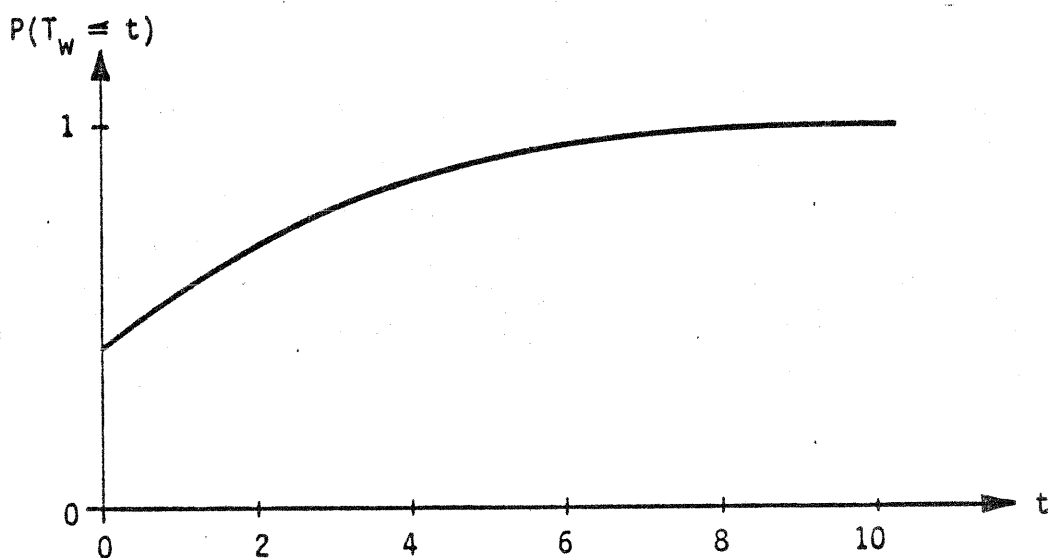


Fig.10. Waiting time d.f. for the queueing system $G/E_k/1, s$.
 (parameters cf. Fig.9)

5. CONCLUSION AND OUTLOOK

We investigated the classical phase concept and found that there are two main problems:

- 1) No efficient method is known to determine the type of general d.f.s when measured values are given.
- 2) The evaluation of the state probabilities for queueing systems with phase-type arrival and service processes may be extremely difficult.

Using the special phase concept with uniform rates of the phases, the number of phases may increase remarkably. However, we found the advantage of this approach to be twofold:

- 1) An efficient algorithm can be formulated for the approximation of measured data with a prescribed accuracy.
- 2) A large amount of computing time and memory can be saved in evaluating the state probabilities and other important performance values.

In this paper we first outlined the approximation procedure. Secondly, we presented results on the performance analysis of $M/G/1$ and $G/E_k/1,s$ queueing systems.

Current and future work concentrates on the area of analysis, where we try to apply the special phase concept to more complex systems with and without priorities.

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