SINGLE SERVER QUEUES WITH GENERAL INTERARRIVAL AND PHASE-TYPE SERVICE TIME DISTRIBUTIONS-COMPUTATIONAL ALGORITHMS

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ABSTRACT

This paper presents a method for the numerical analysis of single server queues with generally distributed interarrival times and a service time distribution of the special phase-type with uniform mean of the phases. Since it was demonstrated in a recent paper that a broad class of distribution functions can be approximated with sufficient accuracy by special phase-type distributions the developed approach represents a widely applicable tool for the numerical analysis of single server queues. The analysis is based on an imbedded Markov chain approach and allows to efficiently compute the interesting performance measures of the queue.

1. INTRODUCTION

The usefulness of single server queues for studying the performance of communication and computer systems has been demonstrated by many investigations. Typical examples are models of central processors or peripheral units in SPC switching systems, models of transmission channels in data communication networks, or models of CPUs and I/O devices within computer systems.

Although an extensive theoretical work has been performed on single server queues (see e.g. /1/ to /4/) it can be generally observed that practical traffic analysts tend to use only those rare results which are analytically simple. For instance, the Pollaczek-Khintchine formula for the mean waiting time in the M/G/1 queue is often applied, even if the assumption of a Poisson input process to the actual system is doubtful. The reason for this is, of course, that for the general G/G/1 queueing system only few results are known, which are in a usable form to obtain numerical results.

This paper addresses the numerical analysis of G/G/1 queues from the following point of view: It is proposed to approximate the given service time distribution function and possibly also the interarrival time distribution function by a general class of phase-type distribution functions (c.f. Section 2). The approximating distribution can be obtained by using an efficient algorithm developed in a former work /5/. This approach has the advantage that the queueing analysis can be performed in a straightforward manner by an imbedded Markov chain approach (c.f. Section 3). The developed way of solution turns out to be an efficient method for calculating the interesting performance measures of the queue: mean waiting time, mean queue length, probability of waiting, and waiting time distribution function in case of first-come, first-served discipline. Numerical results are presented in Section 4 for illustrative purposes.

2. PHASE CONCEPT

The phase-concept, also called method of stages represents a well-known and widely used method in the analysis of queueing systems, see e.g. /6/ to /12/. The principle of this method - developed by A.K. Erlang /13/ - is the notion of decomposing interarrival or service times into exponentially distributed intervals, the so-called phases. Typical members of the family of phase-type distributions are the Erlangian and the Hyperexponential distributions.

Cox /14/ showed that any distribution function having a rational Laplace transformation can be described by the representation shown in Fig. 1, which must be interpreted as follows:

With probability α_0 our random time is zero; with probability $\beta_0\alpha_1$ the time consists of an exponentially distributed phase with mean μ_1^{-1} ; with probability $\beta_0\beta_1\alpha_2$ the time is composed of the sum of two exponentially distributed phases with means μ_1^{-1} and μ_2^{-1} , respectively, etc.

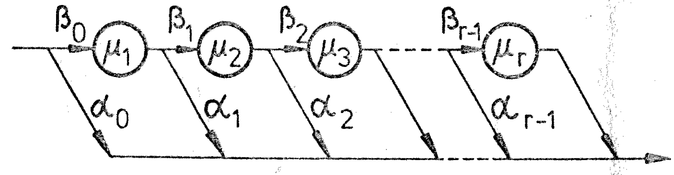

Fig.1. Representation of a distribution function with rational Laplace transform by exponentially distributed phases

The Laplace transform of the corresponding probability density function is therefore given by

$$G_p^*(s) = \alpha_0 + \sum_{i=1}^{r} \beta_0\beta_1\cdots\beta_{i-1}\alpha_i \prod_{j=1}^{i} \frac{\mu_j}{s+\mu_j} \qquad (1)$$

$$(\alpha_r = 1)$$

Since any non-rational function may be approximated arbitrarily closely by rational functions the phase concept can in principle be used for a broad range of applications.

For reasons described below it is advantageous in many cases to restrict ourselves to a subset of all possible phase-type distributions in prescribing a uniform mean (or rate) for all phases:

$$\mu_1 = \ldots = \mu_r = \mu \qquad (2)$$

If we consider for the sake of simplicity only distribution functions with F(0) = 0, i.e. $\beta_0=1$, then the distribution functions for this subset are given by a weighted sum of Erlangian distribution functions

$$F(t) = \sum_{i=1}^{r} q_i\left(1 - e^{-\mu t}\sum_{j=0}^{i-1}\frac{(\mu t)^j}{j!}\right) \qquad t \geq 0 \qquad (3)$$

$$q_i = \beta_1\,\beta_2\,\ldots\,\beta_{i-1}\,\alpha_i \qquad i \in \{1,2,\ldots,r\}$$

Despite the simplification the class of functions in (3) still possesses the following important property:

For any probability distribution function $\Phi(t)$ with $\Phi(0)=0$ for $t<0$, there exists a series of functions (3) which converges weakly to $\Phi(t)$ /10/.

This indicates that the special phase-type distributions with uniform mean of all phases can still be used for approximating general distributions. In Ref. /5/ an approximation algorithm is prescribed which allows to fit with prescribed accuracy the low-order moments and a finite set of values of a given distribution function by such phase-type distributions. The algorithm determines the minimum number of phases r, the branching probabilities $q_1,\ldots,q_r$, and the uniform rate μ of a phase-type distribution with the prescribed properties.

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A second advantage of the special phase-type distributions lies in the fact that the uniform mean property can be efficiently used in the analysis of general queuing systems /2,5,7,8,10/. In Section 3 of the paper it is demonstrated - by analyzing single server queues with general interarrival time distributions and service time distributions of the special phase-type - that we can benefit from this property from a computational point of view, too.

3. ANALYSIS

The system which we study under stationary assumptions is a single server queue with an infinite waiting room. Customers are assumed to arrive with generally distributed and mutually independent interarrival times T_a:

$$P(T_a \leq t) = A(t) = \int_0^t a(\tau)\, d\tau \qquad (4)$$

All customers are served in the order of their arrival (first-come, first-served); their service times T_b are independently drawn from an arbitrary distribution $B(t)$ of the special phase-type:

$$P(T_b \leq t) = B(t) = \sum_{i=1}^{v} b_i \left(1 - e^{-\varepsilon t} \sum_{j=0}^{i-1} \frac{(\varepsilon t)^j}{j!}\right) \qquad (5)$$

We assume that the queue is stable, i.e.

$$\rho = E[T_b]/E[T_a] < 1 \qquad (6)$$

3.2 Imbedded Markov Chain

We investigate the system with the aid of a stochastic process $X(t)$ describing the total number of unfinished service phases in the system. In other words, $X(t)$ corresponds to the sum of all phases pertaining to customers being simultaneously in the queue or in the server. An arriving customer corresponds to a bulk arrival of service phases (c.f. Fig. 2); the distribution of the bulk size is given by the branching probabilities b_i of the service time model.

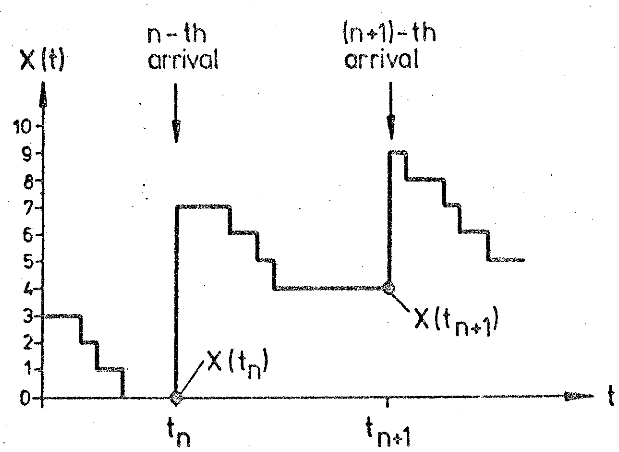

Fig. 2. Number of service phases in the system X(t) versus time t

It is clear that $X(t)$ is not Markovian for general input processes; however, imbedded within $X(t)$ is a Markov process defined at the arrival instants of the customers.

We denote by t_n the arrival time of the n-th customer and by $X(t_n)$ the number of service phases, which the n-th customer finds in the system. Due to the memoryless property of the exponential distribution and the mutual independence of interarrival and service times, $X(t_n)$ forms an imbedded Markov chain.

For first-come, first-served discipline, $X(t_n)$ is equal to the number of service phases customer No. n has to wait; this means that the waiting time distribution can be easily computed if the steady-state probabilities of the Markov chain are known. In what follows we describe an approach for an efficient computation of these probabilities.

3.3 Transition Probabilities

3.3.1 General Interarrival Time Distribution

The one-step transition probabilities of the chain can be determined in a straightforward manner from the number of phases constituting the service time of an arriving customer and the number of service phases which are completed until the arrival of the next customer. The distribution of the former number is given by the branching probabilities b_i of the service time model. The distribution of the latter number can be easily determined: Due to the equal mean property the termination process of the service phases forms a Poisson process, as long as the server is busy.

Therefore, the one-step transition probabilities

$$p_{xy} = P\left(X(t_{n+1}) = y \mid X(t_n) = x\right) \qquad (7)$$

are given by

$$p_{xy} = \begin{cases} \sum_{i=1}^{v} b_i\, e_{x-y+i} & y > 0 \\ 1 - \sum_{i=1}^{v} b_i \sum_{k=0}^{x+i+1} e_k & y = 0 \end{cases} \qquad (8)$$

The quantity e_j is the probability that j exponentially distributed phases with equal rate ε terminate during one interarrival time; it is therefore given by

$$e_j = \begin{cases} 0 & j < 0 \\ \int_0^\infty \frac{(\varepsilon t)^j}{j!} e^{-\varepsilon t} a(t)\, dt & j \geq 0 \end{cases} \qquad (9)$$

For a number of interarrival time distributions the integral in Eq. (9) can be explicitly evaluated; in the following Section 3.3.2 it is shown that for all phase-type distributions - including those with different means of the phases - a very efficient algorithm can be applied for the numerical computation of the probabilities e_j.

3.3.2 General Phase-Type Interarrival Time Distribution

In this section we consider the important case that the interarrival time distribution is of the general phase-type according to Eq. (1) with Laplace transform

$$A^*(s) = \mathcal{L}\{a(t)\} = \sum_{i=1}^{u} a_i \prod_{k=1}^{i} \frac{\lambda_k}{s+\lambda_k} \qquad (10)$$

The generating function of the probabilities e_j is then given by

$$\begin{aligned} e^*(z) &= \sum_{j=0}^{\infty} e_j z^j \\ &= A^*(\varepsilon - \varepsilon z) \\ &= \sum_{i=1}^{u} a_i \prod_{k=1}^{i} \frac{\lambda_k}{\varepsilon - \varepsilon z + \lambda_k} \end{aligned} \qquad (11)$$

Direct inversion of the transform in Eq. (11) yields the following result for the probabilities e_j:

$$\begin{aligned} e_j &= \sum_{i=1}^{u} a_i \sum_{j_1+\ldots+j_i=j} \prod_{k=1}^{i} \left\{\frac{\varepsilon}{\varepsilon+\lambda_k}\right\}^{j_k} \frac{\lambda_k}{\varepsilon+\lambda_k} \\ &= \sum_{i=1}^{u} a_i \prod_{k=1}^{i} \frac{\lambda_k}{\varepsilon+\lambda_k} Q(i,j) \end{aligned} \qquad (12)$$

with $$Q(i,j) = \sum_{j_1+\ldots+j_i=j} \prod_{k=1}^{i} \left\{\frac{\varepsilon}{\varepsilon+\lambda_k}\right\}^{j_k} \qquad (13)$$

The numerical computation of the terms Q(i,j) can be performed in a very efficient way by using the convolutional algorithm, which was developed for the numerical evaluation of queueing networks /15,16/. Applied to our problem the algorithm can be summarized as follows:

(1) for $j \in \{1,2,\ldots\}$ set $Q(0,j) = 0$

(2) for $i \in \{1,\ldots,u\}$ set $Q(i,0) = 1$

(3) for $i \in \{1,\ldots,u\}$ and $j \in \{1,2,\ldots\}$ compute

$$Q(i,j) = Q(i-1,j) + \frac{\varepsilon}{\varepsilon + \lambda_i} Q(i,j-1) \tag{14}$$

By this extremely simple and fast algorithm the probabilities of the Markov chain can be very efficiently evaluated, both from a computing time and storage requirement point of view.

3.4 Steady-State Probabilities of the Markov Chain

3.4.1 Solution of Equilibrium Equations

The stationary probabilities p(x) of the Markov chain are obtained from the following equations:

$$p(y) = \sum_{x=0}^{\infty} p(x)\, p_{xy}$$

$$\sum_{x=0}^{\infty} p(x) = 1 \tag{15}$$

where p(x) is the limiting probability that an arriving customer finds x service phases in the system.

Since we are not able to find a closed form solution of the p(x) for general distributions A(t) and B(t) we retreat to a numerical solution of Eq. (15).

In carrying out the numerical solution we must reduce the state-space of the Markov chain to a finite set. This means we assume that the total number of service phases in the system is limited by a certain value x_m. For the handling of arriving customers which attempt to bring more service phases into the system than there are places available, various strategies can be applied. We used the definition that an arriving customer is not allowed to enter the queue, if not all of his service phases find place in the system. This definition leads to transition probabilities out of states between $x = x_m - v + 1$ and $x = x_m$ which are different from those in Eq. (8), but can also be very efficiently computed using the convolution algorithm /15, 16/. The reason for this choice is that this definition allows moreover to analyze exactly $G/E_k/1,s$ queues with Erlangian service times and a finite waiting room with s places by setting $x_m = (s + 1)k$.

For any definition related to the finiteness of the state-space the strategy for treating queues with infinite waiting room is, of course, the same, namely, to choose x_m large enough that the truncation of the p(x) has virtually no influence on the numerical results obtained.

In order to evaluate the state probabilities p(x) any standard numerical method for the solution of larger systems of linear equations may be applied; our experience showed that the successive overrelaxation method, as described for example in Ref. /4/, represents an efficient, easy-to-program, and numerically stable procedure for our problem.

An additional advantage of this method is that we can benefit from the knowledge of a good approximative solution for the unknowns p(x), by using it as initial values for the iteration (c.f. Section 3.4.2). This often helps to reduce the computation time by an order of magnitude.

3.4.2 Initial Values

Results from a series of tests led us to the following method to find approximate solutions for the state probabilities p(x), which can be used as initial values for the successive overrelaxation method:

If the coefficient of variation of the interarrival times is close to 1 (e.g. between 0.8 and 1.2) we use the state probabilities of an M/G/1 queue with the given service time distribution as an initial solution. How these state probabilities can be efficiently computed is described in Section 3.4.2.1. Otherwise, we use a simple heuristic approach which is described in Section 3.4.2.2.

3.4.2.1 Analysis of M/G/1 Systems

We assume that the service time distribution of the considered M/G/1 system is of the special phase-type according to Eq. (3). We analyze the system by studying the Markov process of the total number of unfinished service phases in the system.

If the arrival rate of the Poisson input process is denoted by λ we obtain the following equilibrium equations for the steady-state probabilities p(x), that x service phases are simultaneously within the system:

$$p(1) = \frac{\lambda}{\varepsilon} p(0) \tag{16}$$

$$p(x+1) = \varepsilon^{-1} \{(\lambda+\varepsilon)p(x) - \sum_{i=0}^{x-1} p(i)\, \lambda\, b_{x-i} \}\ ;\ x \geq 1 \tag{17}$$

For infinite waiting room the probability p(0) is of course given by

$$p(0) = 1 - \frac{\lambda}{\varepsilon} \sum_{i=1}^{v} i\, b_i \tag{18}$$

The most efficient and numerically stable way of computing the probabilities p(x) is to use a similar method as the one proposed in Ref. /12/ for the numerical evaluation of M/G/1 queues using an imbedded Markov chain technique. If we define the auxiliary variables

$$B_i = \sum_{j=i}^{v} \frac{\lambda}{\lambda + \varepsilon} b_j \tag{19}$$

we can rewrite Eqs. (16) and (17) in the following recursive form:

$$p(x+1) = \frac{\lambda + \varepsilon}{\varepsilon} \{B_{x+1}\, p(0) + \sum_{i=1}^{x} B_{x-i+1}\, p(i) \} \tag{20}$$

$$x \geq 0$$

The recursive representation in Eq. (20) has the great advantage that it completely avoids the evaluation of differences which may cause numerical problems if the p(x) were recursively computed according to Eq. (17). Besides the described application to find initial values for the overrelaxation method this approach can also be useful in the numerical evaluation of the waiting time distribution of M/G/1 systems (c.f. Section 3.5).

3.4.2.2 Heuristic Approach

In those cases where the coefficient of variation of the interarrival time distribution is by more than 0.2 different from 1 we use a very simple heuristic approach, which is based on the assumption that the waiting time distribution function of the G/G/1 system is approximately given by

$$P(T_W \leq t) = 1 - W^* \exp(-\frac{W^*}{t_W^*} t) \qquad t \geq 0 \tag{21}$$

This particular form of the waiting time distribution function is achieved if and only if, the state probabilities p(x) are distributed according to (c.f. Section 3.5):

$$p(0) = 1 - W^*$$

$$p(x) = W^* \{1 - \frac{W^*}{\varepsilon t_W^*}\}^{x-1} \frac{W^*}{\varepsilon t_W^*} \qquad x > 0 \tag{22}$$

The parameters W^* and t_W^* are estimated by values for the probability of waiting and mean waiting time, resp., obtained from the approximation formulae for G/G/1 systems given in Ref. /17/.

It turned out that this simple heuristic approach yields an approximation for the state probabilities p(x), which serves over a broad range of parameters as good initial values for speeding the convergence of the successive overrelaxation method.

3.5 Performance Measures

Once the distribution p(x) of the number of phases in the system at the customer arrival instants is known the interesting performance measures of the queue can be computed in a straightforward manner.

The waiting time distribution in case of first-come, first-served is a weighted sum of Erlangian distributions:

$$P(T_W \leq t) = \begin{cases} p(0) & t=0 \\ p(0) + \sum_{x=1}^{\infty} p(x)\{1 - e^{-\varepsilon t} \sum_{j=1}^{x-1} \frac{(\varepsilon t)^j}{j!}\} & t>0 \end{cases} \quad (23)$$

The m-th moment of the waiting time is therefore given by

$$E[T_W^m] = \varepsilon^{-m} \sum_{x=1}^{\infty} \frac{(x+m-1)!}{(x-1)!} p(x) \quad (24)$$

The probability of waiting can be trivially obtained by

$$W = 1 - p(0) \quad (25)$$

The mean queue length is given by

$$\Omega = \varepsilon^{-1} \sum_{x=1}^{\infty} x\, p(x)\, E[T_a]^{-1} \quad (26)$$

For simplicity the upper bounds of the summations in Eqs. (23) to (26) are set to infinity; in practice, the truncation of the state-space is taken into account. The impact of the finite value of x_m can be estimated by computing the (fictitious) loss probability of phases.

4. EXAMPLES

This section presents some numerical results in order to illustrate the range of possible applications of the described analysis.

distribution function	branching probabilities
F_1 (u = 9)	a_1 = 9.138255'-1 a_4 = 1.853350'-2 a_9 = 6.764103'-2
F_2 (u = 18)	a_1 = 7.611742'-1 a_9 = 5.656728'-4 a_{14} = 1.102092'-2 a_{18} = 2.272392'-1
F_3 (u = 40)	a_1 = 7.209440'-1 a_{38} = 2.845652'-3 a_{39} = 5.097046'-2 a_{40} = 2.252398'-1

Table 1. Parameters of the interarrival time distributions F_1, F_2, F_3 of the special phase-type with equal coefficient of variation c_a = 1.802776 .

In our first example we consider a queueing system of the type $G/E_{10}/1$ with an Erlangian service time distribution of order 10 and three different interarrival time distributions F_1, F_2, F_3 of the special phase-type with uniform rate of the phases. Their parameters are given in Table 1; Fig. 3 shows a plot of the three distribution functions. They have been constructed with the aid of the approximation algorithm in /5/ in such a way that all three distributions have equal first and second moments.

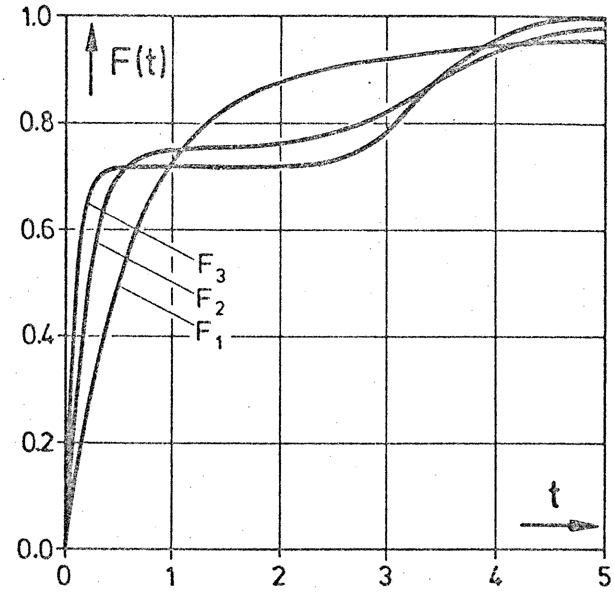

Fig. 3. Interarrival time distribution functions F_1, F_2, F_3 of the special phase-type. All three distributions have equal mean 1.0 and equal coefficient of variation 1.802776 . Parameters c.f. Table 1.

In Figures 4, 5, 6 various performance measures of these three systems are compared. Fig.4 shows the probability of waiting W versus the offered traffic ρ , Fig. 5 shows the mean waiting time t_W relative to the mean service time as a function of the offered traffic ρ , and in Fig.6 the waiting time distribution function is drawn for two different values of the offered traffic ρ = 0.1 and ρ = 0.5. As can be seen from all three figures there exist rather significant differences in the performance values of the three systems.

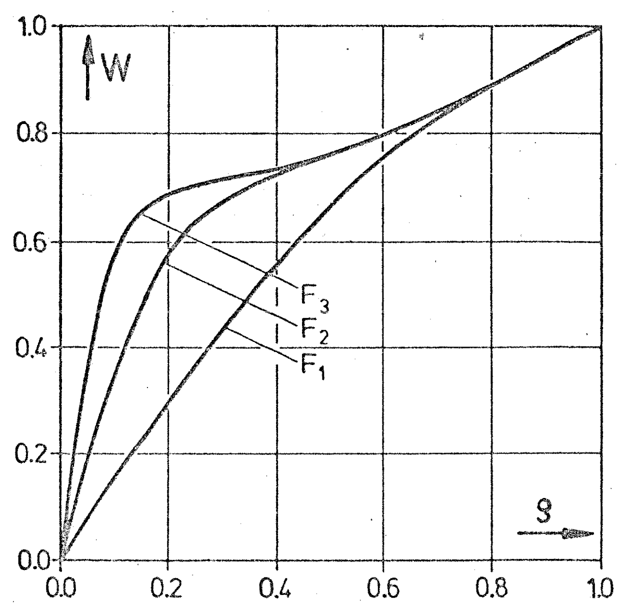

Fig. 4. Probability of waiting W versus offered traffic ρ for the queues $F_1/E_{10}/1$, $F_2/E_{10}/1$, and $F_3/E_{10}/1$. Parameters of the distribution functions F_1, F_2, F_3 c.f. Table 1.

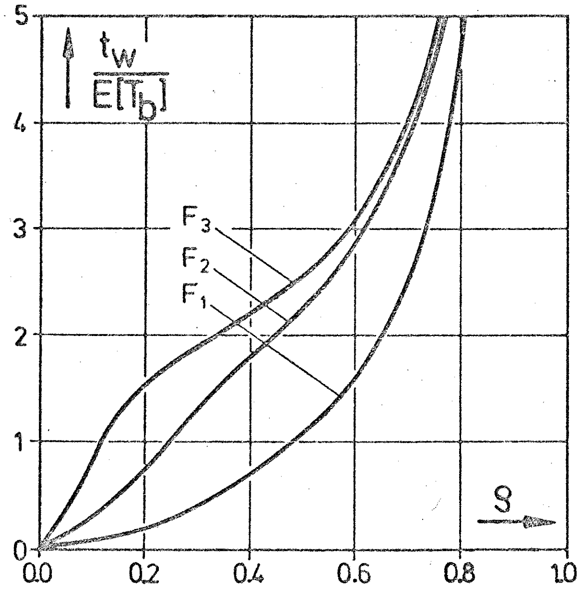

Fig. 5. Mean waiting time t_W relative to the mean service time $E[T_b]$ versus offered traffic ρ for the queues $F_1/E_{10}/1$, $F_2/E_{10}/1$, and $F_3/E_{10}/1$.

Parameters of the distribution functions F_1, F_2, F_3 c.f. Table 1.

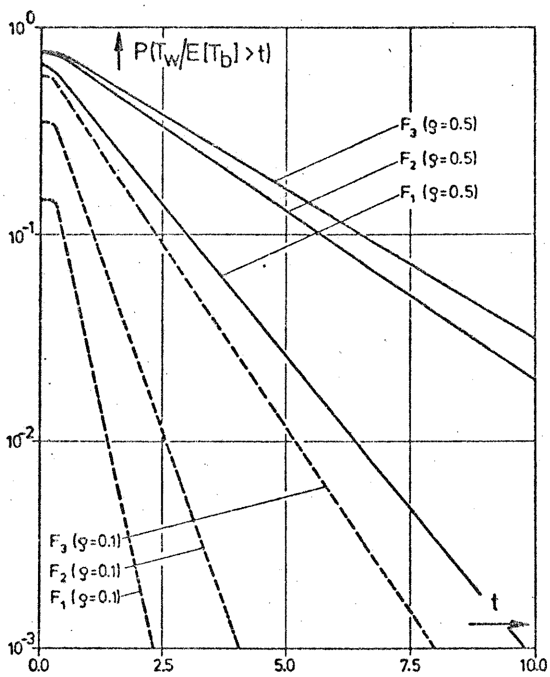

Fig. 6. Waiting time distribution function for the queues $F_1/E_{10}/1$, $F_2/E_{10}/1$, and $F_3/E_{10}/1$

Parameters of the distribution functions F_1, F_2, F_3 c.f. Table 1 .

For our second example we assume a type of service time distribution which is typical of drum and fixed-head disk memories, bold line in Fig. 7. With the aid of the approximation algorithm /5/ a special phase-type distribution F_a has been determined with exactly the same first and second moment and a deviation of the distribution function of less than 0.05 at all integer values of t, dashed line in Fig. 7. The parameters of this approximation are given in Table 2.

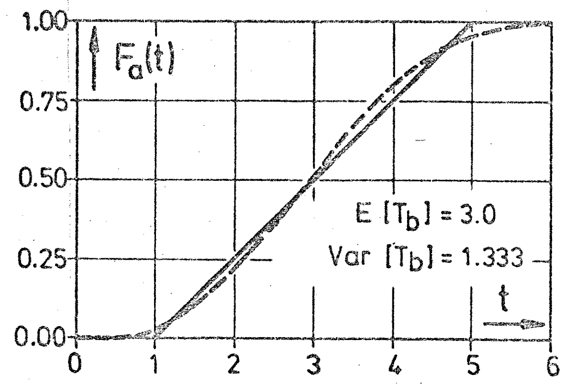

Fig. 7. Service time distribution function for the second example. Parameters of the approximating distributing function (dashed line) c.f. Table 2.

uniform rate of all phases	branching probabilities
ε = 4.618111	b_8 = 1.614637'-1 b_{10} = 2.236070'-1 b_{13} = 3.749187'-2 b_{17} = 5.806837'-1

Table 2. Parameters of the approximating service time distribution function F_a of the special phase-type in Fig. 7.

Figures 8 and 9 show performance measures for a single server queue with this service time distribution and two different types of interarrival time distributions.

The first type is a hyperexponential distribution of order 2:

$$P(T_a \leq t) = 1 - \pi e^{-t/\tau_1} - (1 - \pi)e^{-t/\tau_2} \qquad (27)$$

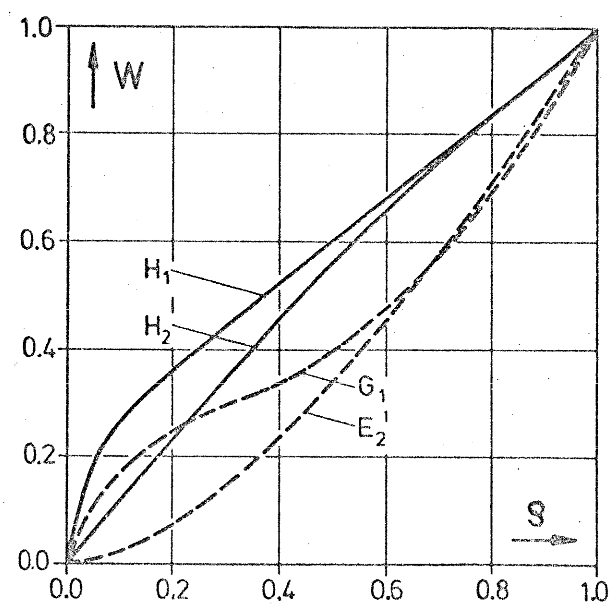

Fig. 8. Probability of waiting W versus offered traffic ϱ for the queues $H_1/F_a/1$, $H_2/F_a/1$, $E_2/F_a/1$, and $G_1/F_a/1$.

Parameters of the distribution functions H_1, H_2, G_1, and F_a c.f. Tables 2, 3 and text.

Two different functions of this type, H_1 and H_2 are considered, both of which have the same squared coefficient of variation $c_a^2 = 1.5$, but differ in the means of the phases τ_1 and τ_2 and the probability π, see Table 3.

distribution function	parameters
H_1	τ_1/τ_2 = 3.6408199'-3 π = 2.4404100'-1
H_2	τ_1/τ_2 = 3.8196517'-1 π = 8.2918000'-1

Table 3. Parameters of the two hyperexponential interarrival time distributions H_1, H_2 with same squared coefficient of variation $c_a^2 = 1.5$.

The second type of interarrival time distribution considered in this example has a squared coefficient of variation $c_a^2 = 0.5$. Again two functions with this property are considered: An Erlangian distribution of order 2 and a special phase-type distribution G_1 with uniform rate of all phases, u = 10 phases, and branching probabilities a_1 = 0.327217 and a_{1n} = 0.672783.

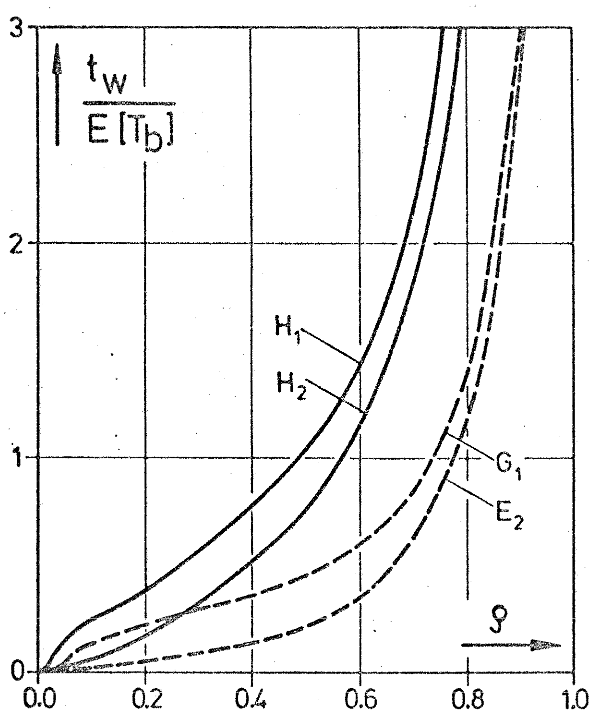

Fig. 9. Mean waiting time t_W relative to the mean service time $E[T_b]$ versus offered traffic ρ for the queues $H_1/F_a/1$, $H_2/F_a/1$, $E_2/F_a/1$, and $G_1/F_a/1$.

Parameters of the distribution functions H_1, H_2, G_1, and F_a c.f. Tables 2, 3, and text.

Both probability of waiting W in Fig.8 and mean waiting time t_W in Fig.9 show again significant differences between systems with identical first and second moment of the interarrival time distribution.

These examples and various other ones indicate that for a generally applicable and fairly accurate numerical analysis of G/G/1 queues not only the low-order moments but also the form of the distribution functions of interarrival and service times should be taken into account.

5. CONCLUSION

The contribution of this paper is to provide a method for the efficient numerical analysis of general single server queues. The presented analysis can be applied if the service time distribution is of the special phase-type with uniform mean of the phases. Since it was demonstrated in a recent paper /5/ that a broad class of distribution functions can be approximated with sufficient accuracy by special phase-type distributions the developed approach represents a widely applicable tool for the numerical evaluation of single server queues.

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