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INTERNATIONAL STANDARDIZING OF LOSS FORMULAE ?

by

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General Remarks

1. Telephone traffic theory has to provide methods for the calculation of connecting arrays. At first the traffic theorist will try to find mathematically exact solutions for his problems. Unfortunately, exact solutions can only be found for rather simple arrays or can't be evaluated even by big computers because of the high rank of the system of linear equations to be solved.

As a matter of fact, connecting arrays now in use are too sophisticated to be calculated exactly (e.g. gradings with more than about 15 outlets).

There is another possibility to get important data such as loss probability and internal blocking : artificial traffic tests on a computer. But this is time consuming and expensive and is usually confined to scientific research.

2. Practical traffic engineering requires simple and easily evaluable methods with sufficient accuracy for planning purposes. Inthe last 1o to 15 years many authors have worked in this field and made useful suggestions. Therefore it seems to be possible now to present the CCI simple calculation methods being suitable for standardization. Thus, a common international basis of comparison could be created which would be useful to all countries. For special and particularlyfor exact investigations Monte-Carlo-methods with artificial traffic will of course still be justified as well as rather complex mathematical solutions.

In the following all given formulae are proposed to be used as standard formulae all over the world. Formulae that can't be derived exactly but only by approximations, have been checked by many traffic tests. We feel that they are best suited for numerous systems with one or more stages, but we will of course accept different solutions, that yield better results.

Programs for evaluation of the given formulae on computers, written in ALGOL, can be made available.

The main purpose of this paper is to start international discussion on this problem.

Standard Formulae

Two types of offered traffic have to be distinguished, depending on the number sources from which the traffic arises :

a. Pure Chance Traffic of Type 1 (PCT1)

An infinite number of sources produces the offered traffic with the mean value of A, each source having an infinitely small call intensity (Poisson Input, according to Erlang's definition).

b. Pure Chance Traffic of Type 2 (PCT2)

A finite number of sources s produces the offered traffic, each idle source having the same call intensity α_o (according to Engset's definition). The mean holding time is assumed to be unity $H_m = 1$.

In both types of traffic, the sources are supposed to be independent from each other and to start calls at random. This implies a negative exponential distribution of idle times of each source. The distribution of holding times is assumed to be negative exponential too.

1. Full-Access Trunkgroups

The probabilities of state p(x) and the congestion B and E resp. can be calculated exactly in full accessible groups with the above mentioned assumptions. No further simplifications have to be made.

a. For PCT1

By means of statistical equilibrium, A. K. Erlang was the first to state the following for the probabilities of state /1/ :

$$p(x) = \frac{\frac{A_o^x}{x!}}{\sum_{i=0}^{n} \frac{A_o^i}{i!}} \qquad (1)$$

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More adequate to evaluation by digital computers is the recurrence formula

$$p(x+1) = \frac{A_o}{x+1} \cdot p(x) \qquad (2)$$

The additional condition $\sum_{x=0}^{n} p(x) = 1$ has to be used.

The probability of loss is given by

$$B_o = p(n) = E_n(A_o) = \frac{\frac{A_o^n}{n!}}{\sum_{i=0}^{n} \frac{A_o^i}{i!}} \qquad (3)$$

Erlang Loss Formula

b. For PCT2

The probabilities of state are given by

$$p(x) = \frac{\binom{s}{x} \alpha_o^x}{\sum_{i=0}^{n} \binom{s}{i} \alpha_o^i} \qquad (4)$$

Erlang's Bernoulli Distribution

or by $$p(x+1) = \frac{s-x}{x+1} \cdot \alpha_o \cdot p(x) \qquad (5)$$

and $$\sum_{x=0}^{n} p(x) = 1$$

Hence the time congestion :

$$E_n(\alpha_o, s) = \frac{\binom{s}{n} \cdot \alpha_o^n}{\sum_{i=0}^{n} \binom{s}{i} \alpha_o^i} \qquad (6)$$

and the call congestion :

$$B_n(\alpha_o, s) = \frac{s-n}{s-Y} \cdot E_n(\alpha_o, s) \qquad (7)$$

2. One-Stage Gradings with Limited Accessibility $k < n$

There have been a lot of different suggestions how to solve this problem.

Because of the high rank of the system of linear equations, exact solutions are possible only for relatively small groups with up to, say, 15 outlets, even with the aid of big computers /2/. Certain simplifications have to be made therefore in order to reduce this high rank.

a. For PCT1

One proposal has been made by A.K. Erlang again /3/, who introduced a special type of grading. In this type of grading (called the "ideal Erlang grading"), the different patterns of distribution of x busy out of n trunks in the group are irrelevant; so the system of 2^n equations is reduced to a system of n+1 equations. As a result, probabilities of state p(x) and probabilitiy of loss B_k can be calculated exactly under the assumption of the ideal Erlang grading (by means of the so-called "Erlang Interconnection Formula EIF").

Unfortunately, this idealized grading is not suited for practical use because of the high number of $\binom{n}{k}$ grading groups.

Gradings with less than $\binom{n}{k}$ grading groups usually tend to higher values of loss, compared with ideal Erlang gradings.

Without further discussing the numerous and interesting different methods for limited-access trunkgroups, we will take up the so-called "Modified Palm-Jacobaeus Formula MPJ" and suggest it for international use. The MPJ has been published at the 3rd International Teletraffic Congress Paris in 1961 /4/ and tabulated in /5/ and /6/. It has now been introduced at the German Post as a standard formula for gradings.

The idea that led to this formula is as follows:

Be given a limited-access trunkgroup with n outlets, accessibility k and the wanted carried traffic Y. If the given trunkgroup with n outlets is imagined to be hunted with full accessibility, a generating offered traffic A_o can be found creating the wanted carried traffic Y and the probabilities of state p(x) according to eq. (1). The following relations hold :

$$A_o = \frac{Y}{1-E_n(A_o)} \qquad (8) \qquad\qquad Y = \sum_{x=0}^{n} x \cdot p(x) \qquad (9)$$

For numerical evaluations, this will require some iterations.

The probabilities of state, creating the wanted carried traffic Y in the imagined full-access group, are now supposed to be a sufficiently accurate approximation for the state probabilities in the actual limited-access group.

We now introduce the probability $\sigma(x)$, that an incoming call will find no free outlet, if x outlets are busy. $\sigma(x)$ can be derived under the assumption, that in the state x all $\binom{n}{x}$ possible patterns have the same probability.

$$\sigma(x) = \frac{\binom{n-k}{x-k}}{\binom{n}{x}} = \frac{\binom{x}{k}}{\binom{n}{k}} \qquad (10)$$

This expression equals exactly Erlang's term in the EIF, derived under the assumption of an ideal grading.

Using the probabilities of state from eq.(1) and the probability $\sigma(x)$ from eq. (10), the blocking probability is:

$$B_k = \sum_{x=k}^{n} p(x) \cdot \sigma(x) \qquad (11)$$

which finally leads to

$$B_k = \frac{E_n(A_o)}{E_{n-k}(A_o)} \qquad (12)$$

Eq. (12) is particularly appropriate for numerical evaluation by computers, since both numerator and denominator can be easily determined by this simple relations: /7/

$$E_{i+1}(A_0) = \frac{A_0 \cdot E_i(A_0)}{i+1+A_0 \cdot E_i(A_0)} \qquad (13)$$

i=o,1, ..,n-1
starting with $E_0(A_0)=1$

At last, the actual traffic A_k offered to the limited-access trunkgroup becomes

$$A_k = \frac{Y}{1-B_k} \qquad (14)$$

Fitting Parameter for different types of gradings.

Obviously, one single formula can't cover all the different types of gradings with their different qualities. As a matter of fact, the MPJ - Formula is valid for inhomogeneous gradings with skipping and proper balancing. If blocking probabilities of other gradings are to be determined, the MPJ - Formula can be easily adapted by the following method of U. Herzog /8/ :

A correcting term ΔA is added to the MPJ-value A_{MPJ} to get the admissible value A_{adm}

$$A_{adm} = A_{MPJ} + \Delta A \qquad (15)$$

The correcting term ΔA can be determined by the empirically found equation :

$$\Delta A = F \cdot \left(\frac{n}{k} - 1\right)^2 \cdot \frac{k-2}{60+4k} \qquad (16)$$

F is called "Fitting Parameter" and is a constant, depending only on the type of grading, not on the probability of loss. This is an advantageous feature of the MPJ-Formula.

F is
- positive for gradings with extremely high grading ratio
- zero for inhomogeneous gradings with skipping and proper balancing
- negative for simplified gradings

For example, the German Post employs the fitting parameter F=-0.3 for simplified but economically designed gradings now in use.

b. For PCT2

The same idea that led to the MPJ Formula is useful in this case, too /6/. For small losses (assuming $A \approx Y$) this idea has been applied in /19/.

The following method holds for all loss-values and is proposed for standardization :

The n outlets of the trunkgroup to be considered are imagined to be hunted with full accessibility. A certain "generating" call intensity α_0 must be chosen to produce the prescribed carried traffic Y.

Hence the probabilities of state are calculated by means of eq. (4) or (5) and α_0 chosen to fulfil the condition

$$Y = \sum_{x=0}^{n} x \cdot p(x)$$

Now, using again the probabilities $\sigma(x)$ according to eq. (1o) one gets

Time Congestion

$$E_k = \sum_{x=k}^{n} p(x) \cdot \sigma(x) \qquad (17)$$

$$= \frac{E_n(\alpha_0, s)}{E_{n-k}(\alpha_0, s-k)} \qquad (18)$$

Call Congestion

$$B_k = \frac{1}{s-Y} \sum_{x=k}^{n} (s-x) \cdot p(x) \cdot \sigma(x) \qquad (19)$$

$$= \frac{B_n(\alpha_0, s)}{B_{n-k}(\alpha_0, s-k)} \quad \text{cf. eq.(7)} \qquad (2o)$$

By analogy with (13), numerical evaluation of eq.(18) can be easily realized by means of the following recurrence relations :

$$E_{i+1}(\alpha_0, s) = \frac{(s-i) \cdot \alpha_0 \cdot E_i(\alpha_0, s)}{i+1+(s-i) \cdot \alpha_0 \cdot E_i(\alpha_0, s)}$$

$$E_{i+1}(\alpha_0, s-k) = \frac{(s-k-i) \cdot \alpha_0 \cdot E_i(\alpha_0, s-k)}{i+1+(s-k-i) \cdot \alpha_0 \cdot E_i(\alpha_0, s-k)} \qquad (21)$$

i=o,1, .. ,n-1
starting with $E_0(\alpha_0, s)=1$
$E_0(\alpha_0, s-k)=1$

The mean offered traffic finally becomes

$$A_k = \frac{Y}{1-B_k} \qquad (14)$$

The actual call intensity per idle source becomes

$$\alpha_k = \frac{A_k}{s-Y} \qquad (22)$$

We have named the new formulae (18) and (2o) "BQ-Formulae", because they consist of a Quotient of Binomial distributions. They have been checked by many artificial traffic tests and found to yield good results.

Just like the MPJ-Formula in case of PCT1 the BQ-Formula can be adapted to simplified gradings of various qualities by a properly chosen Fitting Parameter F.

3. Overflow Problems

Solutions for practical engineering are known only for PCT1 up to now.

If pure chance traffic A is offered to a connecting array with n outgoing trunks, a certain amount R of the offered traffic cannot be handled by the outgoing n trunks. This overflow traffic R has statistical properties different from those of pure chance traffic.

With sufficient accuracy the overflow traffic may be characterized by its mean R plus the additional quantity V (variance) or D=V-R (variance coefficient).

For pure chance traffic : V=R and D=O

For overflow traffic : $V>R$ and $D>0$

In conjunction with overflow traffic the following two problems arise:

a. Calculation of variance V or variance coefficient D of the overflow traffic
b. Calculation of secondary groups (groups to which overflow traffic is offered).

3.1 Full-Access Trunkgroups

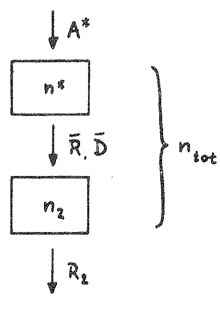
Wilkinson /9/ and Bretschneider /1o/ have studied this problem in detail and derived formulae for variance V and variance coefficient D resp. for the overflow traffic behind a full-access group having n trunks, carried traffic Y and loss probability B :

$$D = R^2 \left[\frac{1}{B(n+1-Y)} - 1 \right] \qquad (23)$$

$$V = D + R \qquad (24)$$

If several overflow traffics from different full-access primary groups are combined, mean $\bar{R}$ and variance coefficient $\bar{D}$ of the resulting overflow traffic can approximately be determined by summing up the single quantities:

$$\bar{R} = \sum_i R_i \quad ; \quad \bar{D} = \sum_i D_i$$



$$R_2 = A^* \cdot E_{n_{tot}}(A^*)$$

$$B_2 = \frac{R_2}{\bar{R}}$$

This resulting overflow traffic $(\bar{R}, \bar{D})$ is now imagined to be generated by a single fictitious full-access group with n^* trunks to which a Poisson traffic A^* is offered.

By connecting the fictitious primary group and the secondary group with n_2 trunks, the secondary group can be calculated by considering the entire group with $n_{tot} = n^* + n_2$ trunks and offered traffic A^*.

Tables or diagrams may simplify the procedure (/9,1o/).

3.2 One Stage Gradings with Limited Accessibility $k < n$

Lotze /11/ has published a method for calculating the variance coefficient of the overflow traffic behind gradings. He takes advantage of the fact, that the overflow traffic behind one single grading group has about the same statistical properties as the overflow traffic behind a full-access group with k lines and having the same probability of loss as the whole grading. In addition, the correlation between the partial overflow traffics of the grading groups has been taken into account. The treatment in detail /19/, /11/ yields for the variance coefficient

$$D = p \cdot R^2 \cdot \frac{k_1}{n_1} \cdot \left[1 + \frac{1}{2 g_1} \left(\frac{n_1}{k_1} - 1 \right) \right] \qquad (25)$$

where

n_1 number of trunks in the primary grading
k_1 accessibility in the primary grading
g_1 number of grading groups in the primary grading
p peakedness parameter

$$p = \frac{1}{B_k (k_1 + 1 - A_0 (1 - B_k))} - 1 \qquad (26)$$

B_k probability of loss in the primary grading
A_0 fictitious traffic offered to a full-access group with $n = k_1$ lines, leading to a congestion $B = B_k$

The calculation of secondary gradings is an extension of the method in 3.1 :

The overflow traffics from several primary gradings are combined, the resulting overflow traffic with mean $\bar{R} = \sum R_i$ and variance coefficient $\bar{D} = \sum D_i$ is supposed to be generated by a single fictitious primary grading (so-called "equivalent primary grading" with n^* trunks and accessibility k^*) to which a fictitious pure chance traffic A^* is offered.

Now the equivalent primary grading and the secondary grading to be calculated (n_2, k_2) are considered to be one grading (with $n^* + n_2$ trunks, accessibility $k^* + k_2$, offered traffic A^*). As a supplementary condition, the quantities n^*, k^* of the equivalent primary grading must be chosen such that the grading as a whole $(n^* + n_2, k^* + k_2)$ is inhomogeneous and reasonably designed.

By means of the tables in /13/, the number of trunks n2 for a given loss B_2 can be easily found. It holds : $n_2 = n_{2RANDOM} + \Delta n_2$

where $\Delta n_2 = \frac{\bar{D}}{\bar{R}} \cdot \{ C_1 \cdot (\bar{R} - 20) + C_2 \}$

The functions $C_{1,2} = f(B_2, k_2)$ can be read out of diagrams in /13/.

Moreover, the loss B_2 for given number n_2 can be determined by the tables in part B of /13/. The application of this method to the economic design of networks with alternate routing is handled in detail in this book by R. Schehrer /12/.

3.3 Link Systems

By analogy with 3.2 a method can be developped that holds for link systems of two or more stages. This problem will be dealt with in a separate paper presented to the 5.ITC by Herzog /15/.

4. Mixed External and Internal Traffic

Mixed external and internal traffic arises, if a switching stage handles traffic in outgoing as well as incoming direction (both-way switching stage).

4.1 Full-Access Trunkgroups

Probabilities of state p(x) and congestion can be determined exactly in full-access trunkgroups /16,17,18/.

a. For PCT1

$$p(x) = \frac{\sum_{r=0}^{[\frac{x}{2}]} \frac{A_i^{\,r}\, A_e^{\,x-2r}}{r!\,(x-2r)!}}{\sum_{x=0}^{n} \sum_{r=0}^{[\frac{x}{2}]} \frac{A_i^{\,r}\, A_e^{\,x-2r}}{r!\,(x-2r)!}} \qquad [\tfrac{x}{2}] = \text{entier}\,(\tfrac{x}{2}) \qquad (27)$$

x=o...n

The corresponding recurrence relation for easy computation is

$$p(x+2) = \frac{A_e}{x+2}\cdot p(x+1) + \frac{2A_i}{x+2}\cdot p(x) \qquad (28)$$

x=o...n-2

Additional conditions: $p(1) = A_e \cdot p(0)$

$\sum p(x) = 1$

Hence the probability of loss :

$$B_0 = p(n) + \frac{A_i}{A_e + A_i}\cdot p(n-1) \qquad (29)$$

b. For PCT2

If a single type of sources s is assumed, generating external as well as internal traffic, the following recurrence formula yields the probabilities of state :

$$p(x+2) = \frac{s-x-1}{x+2}\cdot \alpha_e \cdot p(x+1) + \frac{s-x}{x+2}\cdot 2\alpha_i \cdot p(x) \qquad (30)$$

x=o...n-2

Additional conditions: $p(1) = s\cdot\alpha_e\cdot p(0)$

$\sum p(x) = 1$

Hence the time congestion:

$$E_n = p(n) + \frac{\alpha_i}{\alpha_e + \alpha_i}\cdot p(n-1) \qquad (31)$$

and the call congestion:

$$B_n = \frac{s-n}{s-Y}\cdot p(n) + \frac{\alpha_i}{\alpha_e+\alpha_i}\cdot\frac{s-n+1}{s-Y}\cdot p(n-1) \qquad (32)$$

4.2 One-Stage Gradings with Limited Accessibility $k < n$

a. For PCT1

According to the MPJ-Formula, the distribution of a full-access trunkgroup (27) is used, the offered traffic A_{oe}, A_{oi} chosen properly to produce the wanted carried traffic Y_e, Y_i (iteration necessary).

If we again take advantage of the probability $\sigma(x)$, eq. (1o), the probability of loss turns out to be /18/ :

$$B_k = \sum_{x=k}^{n} p(x)\cdot\sigma(x) + \frac{A_i}{A_e+A_i}\cdot\sum_{k-1}^{n-1} p(x)\cdot[1-\sigma(x)]\cdot\sigma(x+1) \qquad (33)$$

b. For PCT2

According to the BQ-Formula, the distribution p(x) of a full-access trunkgroup (3o) is used, the offered traffic intensities α_{oe}, α_{oi} chosen properly to produce the wanted carried traffic Y_e, Y_i (iteration necessary).

Hence the time congestion:

$$E_k = \sum_{x=k}^{n} p(x)\cdot\sigma(x) + \frac{\alpha_i}{\alpha_e+\alpha_i}\cdot\sum_{k-1}^{n-1} p(x)\cdot[1-\sigma(x)]\cdot\sigma(x+1) \qquad (34)$$

and the call congestion

$$B_k = \frac{1}{s-Y}\cdot\sum_{x=k}^{n}(s-x)\cdot p(x)\cdot\sigma(x) + \frac{\alpha_i}{\alpha_e+\alpha_i}\cdot\frac{1}{s-Y}\cdot\sum_{k-1}^{n-1}(s-x)\cdot p(x)\cdot[1-\sigma(x)]\cdot\sigma(x+1) \qquad (35)$$

List of Used Symbols

n	number of circuits in a trunk group
k	accessibility in a grading
g	number of grading groups
s	number of traffic sources
A	mean value of offered traffic intensity (in Erlang)
Y	mean value of carried traffic intensity (in Erlang)
x	number of trunks occupied
p(x)	probability of x trunks being occupied
B	probability of loss (call congestion)
E	time congestion
σ(x)	probability, that a call will find no free outlet, if x outlets are busy
F	fitting parameter
V	variance
D	variance coefficient

Indices:

o,n	for full-access trunkgroups
k	for limited-access trunkgroups
e	external
i	internal
1	primary
2	secondary

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